Managerial Issues of Teaching Mathematics

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Abstract: Mathematics belongs among the least popular subjects. The reasons are quite known and well described for example in [1]. For that reason, we concentrate on methods which could improve its image among general audience. Instead of talking about internal problems of mathematics, our paper discusses the issue as a managerial problem. We use an analogy: *What has to be done if Teaching Mathematics would be an enterprise with a similarly bad reputation?* Based on the parallel, we propose long-term solutions and later exemplify them. As shown, the most of them requires substantial changes in mathematicians' minds and in their approach to their teaching methodologies.

1. Introduction

Let us discuss the case of a company X:

- *X*'s public reputation of is very bad.
- Just a few employees of X care about improving its picture.

One does not need to be a businessman or economist to guess that *X* faces a gloomy future.
Teaching mathematics satisfies both above criteria:

- The public reputation of *Teaching Mathematics* is very bad.
- Not so many individuals try to improve the popular picture.

Based on the analogy, we can conclude: if there would be an enterprise named *Teaching Mathematics*, it would have probably already crashed and *Mathematics* disappeared from many schools. We have to ask ourselves:

- Is there precedence to such a case?
- Can we prevent its repetition?

A precedent exists. The *Latin* language was the lingua franca of scientists during the Medieval Ages and was taught as a compulsory subject at high schools long after – in Central European countries for a big portion of 20th century. In a stepwise manner, its dominant position has been diminished. Now it occasionally appears in the form of an elective. Such a limited offer seems to fully satisfy its market's demands. Voices for its return to a more significant position practically do not exist.

Mathematics teachers and enthusiasts should therefore ask themselves: *Can we prevent Mathematics to become a follower of Latin?* Due to his background as a Professor at the Management College in Bratislava and the author of the course *People, Technology and Management* for the Laureate Program of the University of Liverpool, the author proposes his response based on a managerial point of view. At first glance, making interconnections between Mathematics and Management may look strange. For that reason, our first aim is to convince our reader that there are good reasons to do so. Above we stated: "*There is no public demand for the return of Latin*". Evidently, if parents would feel the need, they would form a strong pressure to the establishment to introduce it – and the authorities would obey. The present situation of Latin is the result of non-existence of such pressure – a typical “demand-offer” relationship well-known by salesmen.

In our argumentation we are looking for additional relations, too. Below we therefore consider *Teaching Mathematics* as a (virtual) company and try finding answers to the principal questions:

- What’s wrong with Teaching Mathematics that makes its reputation so negative?
- What business principles can propose ways out?
- What does their application mean for our community?

The last question is demonstrated by a series of examples which are “purely mathematical” despite the fact that they have been inspired by our managerial considerations. In such a way they illustrate that a bridge between the disciplines exists and can be efficiently applied for benefits of the both.

2. What Danger Do We Face?

We start with using managerial terminology for clarifying the subject of our discussion. When people talk about “bad reputation of *Mathematics*”, their criticism may address either science itself or its teaching in schools. In our opinion, only the latter is correct. Those who deal with Mathematics as professionals know its importance well. The critics are predominantly grouped from people who have got just a limited contact with it during their school education. Despite the fact that throughout their adult lives they have a little to do with it, their bad feelings and attitude persist.

From the managerial point of view, *Mathematics* is in the position of the parent company; *Teaching Mathematics* is its daughter. The verification of this statement is quite obvious. If there would not be *Mathematics*, there would be no *Teaching Mathematics*. (If there would be no car producer, car dealers could not exist – they distribute what was produced by others.)

As usual, the dependence is reciprocal. Car dealers cannot exist without car production, but car production becomes meaningless without cars’ sale. Similarly, there would be no mathematics education without mathematics but mathematics would face the extinction without (some extension of) teaching it. Compare the situation with *Latin*. It is still taught at a limited scale – just sufficient for “keeping it alive”. Similarly, *Teaching Mathematics* would unlikely fade away entirely. It would persist at a limited scale necessary for its survival.

For that reason, our questions do not address teaching “*Mathematics for its followers*”. In this form, it will survive anyway. What we do face is a danger of extinction of teaching “*Mathematics for others*”. Our Latin lesson says: “To prevent the disappearance, concentrate on creating demand. The greater is public demand, the lower becomes the probability of extermination.” While forming demand, we have to prioritize *Mathematics for others*. Danger coming from this “market segment” is more imminent as this is the group of “potential troublemakers”. If we win their endorsement, the danger of extinction vanishes.

Unfortunately, the group is specified by exclusion. Consequently, it is not a homogenous body. For that reason, a single strategy will hardly work – the approach described in our paper should be considered as an example as well as our call for alternative ones. Their combination could lead to a better image of Mathematics education in eyes of public.

3. Setting up Priorities

As explained above, the relationship between *Mathematics* and *Teaching Mathematics* resembles that between "parent" and "daughter" companies. The parent establishes its daughter(s) for special purposes: banks set up leasing companies or assets management organizations; car dealers establish brand-oriented dealers or spare part producers. The daughters allow their mothers concentrate on their core activities. *Teaching Mathematics* releases its mother from the need to solve daily tasks of spreading its earlier discoveries among population. As a result, the mother can concentrate on exploring and finding new knowledge.

As managers, we may then ask: *Does Teaching Mathematics distinguish itself enough from its mother - Mathematics?* Again, an analogy is handy. Everyone likely agrees that everyday functions and work organization of car dealers substantially differ from that of producers. Similarly, in our paper we discuss the activities of *Mathematics for others* in the maximum degree of isolation from its mother. We pose ourselves into the role of a management that designs its new daughter. Our aim is to establish it in a manner that will help to maximize benefits for both mother and daughter and to minimize unnecessary overlapping between them – we will form the "flesh and spirit" of the new body. Tasks of this sort are solved during the strategic planning – the function each company must regularly execute every few years. So, even if *Teaching Mathematics* exists for centuries, its aims and methods should be revised and redesigned at regular intervals. We can therefore regard our task as another stage of this standard process – the moment when we analyze whether or not there are reasons for building a new daughter or changing its methods of operation.

Wikipedia [2] characterizes Strategic Planning in the following manner: *Strategic Planning is the formal consideration of an organization's future course. All strategic planning deals with at least one of three key questions:*

"What do we do?"
"For whom do we do it?"
"How do we excel?"

4.1 What do we do?

The first question targets the current state of the organization. It should take into account not only the subject of the company's interest but also effectiveness of its procedures and quality of its output. Many analyses documented the inferiority of mathematical education for our market segment – for "all others". It is often expressed by the statement "I hate Mathematics" [1]. The reason for the loathing comes from its most frequently applied method: "*Repeat what mathematicians do and you will (hopefully) start understanding Math*". In principle, "learning by doing" is not a bad idea and it functions in many cases. At the same time, it presumes that the learners are capable of understanding the output of their efforts and can evaluate its correctness and quality. This is important for building the feedback that enhances learners' further progress.

The misconception is somewhere else. The execution of formal manipulations – the simplification of complex expressions or construction of geometric bodies – is not the primary task of mathematicians. They first analyze the problem in their minds. (If) they get a promising hypothesis, they try to verify it. Using formal algebraic manipulations and geometric constructions is therefore a supportive method, not their main aim. Traditional *Teaching Mathematics* went from this concept too far. Too much accent is done to their performance changes learning to drill. Drill prohibits understanding.

So, the lesson from the first question should be: *Formal algebraic manipulations and geometric constructions themselves are not the best way to understanding Mathematics. If we want to simulate the work of mathematicians in a more realistic manner, we should concentrate on formulation of problems, forming hypotheses and verification of solutions.*

4.2 For whom do we do it?

The second question addresses another key aim. The target groups of *Mathematics* and *Mathematics for all others* radically differ. The former one is formed of professional mathematicians and active users of Mathematics (engineers, designers, physicists, biologists, and other specialists) who need new or improved methods for solving their problems. The latter one consists of much more general audience. As our intention is to increase the demand, we have to demonstrate the usefulness of mathematics to them.

The "usefulness" can be expressed by various means: practicality, beauty, joy, etc. In fact, many mathematicians and teachers of mathematics claim that they are familiar with them and try to implement then into their classrooms. The question is whether their perception of "beauty" coincides with that of their pupils and students. For example, everyone will agree that the fractal in Figure 1 (taken from [3]) is beautiful but people may argue what makes it nice. Mathematicians will probably prefer the possibility to define it by a formula – and to deform and modify it by entering different parameters. To artists, similar characterization of beauty will likely sound strange. To minimize misunderstandings, mathematicians should allow the artists to manipulate with parameters and let them to feel their influence to the form and the size of the picture. Later, they could ask them to predict forthcoming modifications caused by changing parameter values. Notice that even without knowing the formulas both groups are coming much closer to "genuine mathematical activities" – forming hypothesis and verifying them than with a traditional classroom.

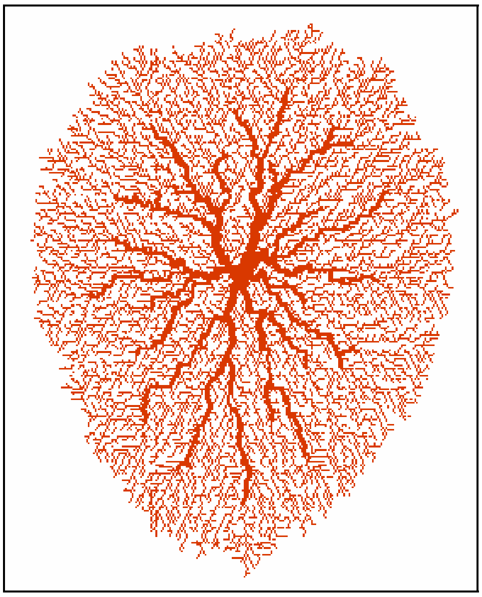

Figure 1. *Fractal*

Using fractals and other complex structures, we can also demonstrate the general character of mathematics. Ask several people what the object resembles – their answers will differ. It could be a tree viewed from above, a coral, a river system ending in a lake in the middle of the picture, windpipes in lungs, and others. All of them are defined by the same formula!

A lot about Mathematics can be told and explained without routine calculations and formula memorizing. It does not mean that we entirely reject teaching them. We just underline that their role seems to be exaggerated. Our second lesson sounds: *For the most of population, it is difficult to understand raison d'être of mathematical concepts, formulas, expressions, and manipulations unless they assign meanings to them. Individual interpretations do not need to be unified – each person may have his/her own.*

This idea explains why even illiterate persons can perform four basic arithmetic operations – they understand their meaning and importance. Troubles start when similar connotations are lost.

4.2 How do we excel?

The third question addresses the role of the teacher and its specifics. As this conference is oriented to Computers in Education, we should likely prefer methods intensively exploiting IT but there can be other parallel methods of education with different accents.

What then appears is the problem of compatibility of these different streams. Contemporary mathematics is extremely large. It consists of many fields, each of them with its specific problem-solving methods, terminology and results. Professionals in one field have often problems to understand those belonging to other ones. Designers of different courses have to decide which fields to select and why. Again, the situation resembles that of car dealers. No one sells all cars. They specialize to certain brands, categories and price intervals. This implies that *Teaching Mathematics* may develop into a series simultaneously acting “companies” each of them possibly “selling” different products. Some branches will be relatively independent e.g. graph theory or logic. The same holds for their problem-solving methods. Proofs in *Logic* are very formal; proofs in *Graph Theory* often use pictures and intuition. The methods applied in Logic are appropriate for *Mathematics for followers*, whilst Graph Theory fits well to *Mathematics for others*.

The “sister companies” – alternative branches of *Teaching Mathematics* – that will stress them have to be (to a certain) degree independent. At the same time, all cars are “compatible” in the meaning that driving skills learned using a particular car can be easily transformed to any other. So, a driver can change a car without necessity to be retrained. Designers can learn from this analogy, too. Their course content does not need to be identical, learning styles may substantially differ. However, there must remain something that helps characterizing all of them as “mathematically associated”. The most affordable way of achieving such alternatives is a stepwise evolution from the traditional “Arithmetic – Algebra – Calculus” approach. A high proficiency in formula manipulations was a necessary pre-condition for solving equations and inequalities as well as a prerequisite for Calculus. Today, Computer Algebra Systems perform these operations much faster and with higher precision. Much less time needs to be spent on drilling them. The saved time can be devoted to activities that are more important to “non-followers” and more welcome by them. Thus, our last lesson says: *Release your students from drill. Give them opportunity to appreciate the core of mathematics in a very individual way whenever possible designed directly for them.*

Some of these alternative branches can target quite narrow “market segments”. Similar moves have already been done. For example, Haapasalo and his colleagues wrote alternative textbooks [4], [5] and [6] for vocational schools primarily based on the concepts needed in their particular professions. In next chapters we show an approach based on separation of the roles between learners and information technology. While IT is much better in performing any sort of standard procedures, humans excel in considerations and putting things into the context. We try to intensify the separation of their functions and develop our student’s intuition in recognizing mathematical aspects of reality.

To close our trip to *Management*, we can conclude that results are not entirely revolutionary. Most of them are known for long:

- Building relationships between mathematics and reality;
- Student-centered problem-solving approach;
- Satisfying individual interests and needs of learners.

What is new is the fact that – in order to satisfy its clients’ needs – *Teaching Mathematics* must place them as its priorities. Its mathematical content must become just an “integrated part of the game”. The current formal content of classes must be (intelli)gently covered by means that make it attractive. In our paper, we focus on how to make it in a user-friendly, “edible” and “digestible” way.

4. Making Mathematics More User-friendly, Edible and Digestible

4.1Mathematicians as Commons

As a part of a CASIO FX-9860G project supervised by Wei-Chi Yang (and later edited by L. Paditz in [7]), the author worked on a series of educational programs aimed to help understanding mathematical concepts, notions and procedures. Hyperbolic functions were among his assigned tasks. Suddenly, the author realized that what he learn during his university study were isolated facts like their definitions

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

and their derivation rules. He had no idea how the curves look and whether they appear in reality (and if so, where). His knowledge was based on memorizing, not comprehension. As his perception of mathematics has changed since his graduation, he wished to form teaching material in accordance to the above principles.

Luckily, at the present time, teachers and students can be better informed than it was typical for the period of my study. The Internet pages offer relevant and reliable information. Among many pages referring to hyperbolic functions, three [8, 9, 10] provided enough data for the author to accomplish his task.

Above formulas describing hyperbolic functions can hardly attract attention of a common learner. We recommend an opposite approach. One should propose simple problems which produce hyperbolic functions as their results. There is such a problem: *Hold the ends of a rope in your hands. The hanging rope forms a curve. What curve is it?*

Many people wrongly guess that the curve (named *catenary*) is a parabola. Even famous mathematicians (including Galileo Galilei, at about 1600) made the identical mistake. In 1669, Jungius found the right formula. The curve is cosine hyperbolic. It can be drawn using a graphing calculator – see Figure 2.

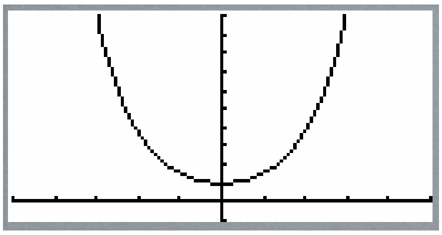

Figure 2. *Catenary*

We suggest teachers telling to their students about Galileo’s mistake. Mathematicians are often taken for as people never making errors. Showing a fault of the well-known personality can make him more human, more similar to “all of us”. We believe that frequent confessions of the movie stars and popular singers: “I was always bad in mathematics” originate in their desire to express: “I am one of you, I am not perfect”. Pointing to errors of top scientists may serve to a similar purpose. Mathematicians are people like any others. It is quite strange if for example biologists or geologists are appreciated as common humans, but mathematicians are not. While this picture in human minds persists, the negative (or better saying repelling and discouraging) picture of mathematicians will persist.

4.2 Visualization of Formulas

During the next step, the students can simulate the process of changing the shape of the curve by moving their hands closer (farther) to (from) each other. The formula of parameterized version of the cosine hyperbolic looks as follows

$$y = a * \cosh(x / a) = 0{,}5 * a * (e^{\frac{x}{a}} + e^{\frac{-x}{a}})$$

The learners equipped with advanced calculators with built-in hyperbolic functions do not need to know the formula. To perform simulations, they simply type them with varying coefficients. In the case shown in Figure 3, the parameter *a* changes from 1 to 4. Figure 4 shows the result. It simulates stretching hands with the rope – the minimum of consecutive curves goes higher and higher.

Figure 3. *Formulas with 4 different parameters*

Notice that we could create the identical picture using different ways. Our example uses the "pure" cosine hyperbolic:

y_1 = cosh(x)
*y_2 = 2*cosh(x/2)*
*y_3 = 3*cosh(x/3)*
*y_2 = 4*cosh(x/4)*

The other method may apply more complex formulas with exponentials:

$$y_1 = \frac{1}{2} * (e^{x} + e^{-x})$$

$$y_2 = e^{\frac{x}{2}} + e^{-\frac{x}{2}}$$

$$y_3 = \frac{3}{2} * (e^{\frac{x}{3}} + e^{-\frac{x}{3}})$$

$$y_4 = 2 * (e^{\frac{x}{4}} + e^{-\frac{x}{4}})$$

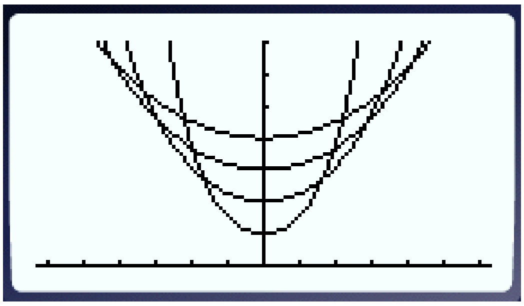

Figure 4. *Stretching hands with the rope*

There is no reason to introduce the latter set in *Mathematics for others* unless the teacher want to point to the fact that the same function(s) can be specified in different ways.

Is there any reason to give a preference to the notation with exponentials? It for sure existed in times of tables of logarithms and exponentials. The needed value did not need to be calculated – it could be found in the book. So, the formula could be manually evaluated much faster. In the Computer Age, this argument is no more valid. If its language allows typing the expression with cosine hyperbolic, the calculations are virtually equivalent. One the other hand, the formulas with *cosh* are more legible and express teacher's intention better.

4.3 Assigning Meaning to Abstract Objects

There are other applications of cosine hyperbolic. The website [9] contains an excellent animation showing regular polygons smoothly rolling on rails formed from identical sections of cosine hyperbolic – see Figure 5. The learners can discuss the application of the principle. In accordance to our experience, their views differ quite radically – from a railway with polygonal “wheels” appropriate for mountain cable railways to in-time delivery in which the number of polygon sides corresponds to the number of deliveries per constant period and the rail to the production process. Their conclusion is: *The more frequent deliveries, the smoother production.*

Even if some of these interpretations might not be entirely correct, there are not many reasons to forge learners to put them into accordance with the theory. In our opinion, building their own perception of the problem and starting understanding that it can be characterized by means of mathematics is more important that the making their interpretations perfect.

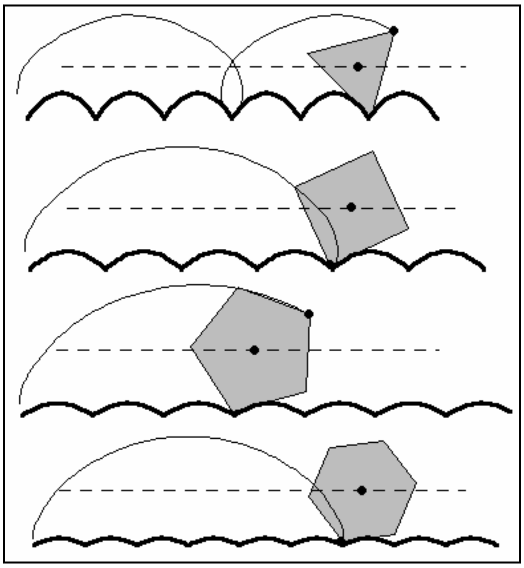

Figure 5. *Rails from sections of cosine hyperbolic*

4.4 Rising Students’ Interest in Solving Mathematical Problems

The previous sections might create a false impression that we are against any elements of “traditional” education which includes calculations, manipulation with expressions and geometric constructions. Not at all. We only wish to put them into a real-life context and create a more pragmatic picture of mathematics and mathematicians. Calculations are the heart of mathematics but again they should not be “calculations for sake of calculations”. They should be interconnected with reality whenever possible. Our next example shows a way.

In 1965 in Saint Louis (USA), a large arch has been built. The following formula [10] describes it:

`y = 693.8597 – 68.7672*cosh(0.0100333x)`

The teacher can ask the students first to find its photos in order to realize its magnitude and beauty. After showing them, the questions like: *How high is the arch? How far is each pillar from the other?* will become natural consequences of their findings. Using computer graphics, they can also draw its model – Figure 6.

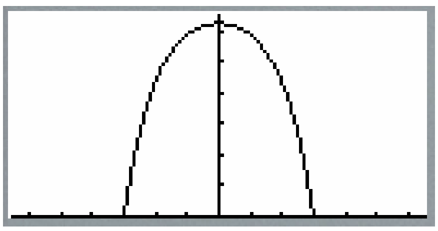

Figure 6. *Drawing Saint Louis Arch*

There are two ways of calculating the height:

1. The graph shows, that the highest point corresponds to x = 0. Taking this as a fact, we get the height as the result of calculating 693.8597 – 68.7672*cosh(0). With an advanced calculator, easily produces the result 625,09. Americans calculate in feet, so the result is to be converted into 190,5 meters.

2. As our eyes may cheat us (and x = 0 does not need to correspond to the maximum), calculating the exact maximum is another method. Presuming that the maximum is evidently somewhere in the interval [-10; 10], we can order its direct calculation. In such a case one can benefit from the FMAX function calculating the maximum of its argument with a given interval. The result is identical.

Figure 2 also shows that the pillars touch the ground in the points having their y-coordinate equal to zero. The students easily conclude that the bases are the roots of equation

```
693.8597 – 68.7672*cosh(0.0100333 x) = 0
```

This – otherwise extremely complex – calculation can be quickly solved using an advanced calculator. The roots are -299.226 and 299.226 (in meters –91 a 91) so the distance between pillars is 182 m.

Notice that even many university courses do not teach to solve similar problems due to their complexity. Manual calculations "must" end up with linear and quadratic equations. Problems with linear (or at most quadratic) solutions were preferred because the use of manual calculations prohibited finding their solution(s) in affordable time. With advanced calculators, all equations become "equally difficult". There is no need for favoring problems with simple (one-digit) coefficients and integer solutions. This opens doors for introducing more realistic problems into classrooms.

Another benefit of the proposed approach lies in its friendliness to the "user". Learners are not repelled from complex mathematics by a necessity to make long and boring calculations. Instead of them, they can focus on grasping principal concepts like the "maximum of a function", "value of a function in a point of our special interest", "roots", etc. Also, the formulation of problems is more "digestible". Traditional courses would ask the students:

- Find the maximum of the function `y = 693.8597 – 68.7672*cosh(0.0100333x)`;
- Calculate the roots of the equation `693.8597 – 68.7672*cosh(0.0100333 x) = 0`.

We prefer questions: *How high is the arch? How far is each pillar from the other?* Notice that the problems are identical – just the latter formulations look more human.

4.5 Imperfect Calculations for Our Imperfect World

There is an elevator inside a pillar of the Saint Louis Arch that transports visitors up to the viewpoint at the arch top. *What distance do the passengers travel?*

The direct method requires using calculus. However, one can calculate an approximate value using a spreadsheet. From our above calculations we know that the topmost point of the arch has coordinates (0; 625) and the right base (299; 0). One can draw a direct line between them. It represents the shortest possible lift between the considered points. Pythagoras Law can be used for calculating its length – 693 feet, i.e. 211,2 m.

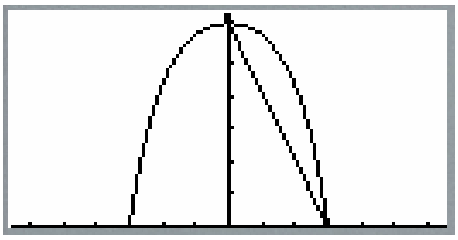

Figure 7. *The shortest possible lift*

Because the lift is located inside the arch, it is in fact longer. One can see in Figure 7. A closer value can be calculated by splitting the straight line into two (touching the arch also in x = 150). One can easily calculate (150; 531,35) as the coordinates of the point. (That's why we stressed above "calculations related to the points of our special interest".)

The arch curve is than approximated by two broken lines: the first from (0; 625.0925) to (150; 531.35), the second from (150; 531.35) a (299.226; 0). Again, using Pythagoras Law give us the result 728.79 feet, i.e. 222.14 m. Everyone sees a substantial difference. So, the new result is a significant improvement.

By increasing the number of sections, we can reach even better approximations. Let us divide it for example using six dividing points (for x = 0, 50, 100, 150, 200, 250, 299.226 feet). The calculation can be speed up using a spreadsheet calculator. Figure 8 shows a method:

SHEE	A	B	C	D
1	x	y	Part	Total
2	0	625.09		738.9
3	50	616.25	50.774	
4	100	587.47	57.691	
5	150	531.35	75.162	
6	200	433.47	109.91	
7	250	268.68	172.21	
8	299.22	7.8E-4	273.15	

Figure 8. *Using spreadsheet calculations*

- The column A contains the x-coordinates of the dividing points.
- The values in the column B are the values of the function defining the arch. First we write

  = 693.8597 – 68.7672 * cosh(0.0100333 * A2)

  and then copy it into the remaining cells of the column B.
- In the column C we calculate the distance between two consecutive points. Again, one formula must be typed but the remaining can simply be copied.
- Their sum is in the cell D2. It is 738,9 feet i.e. 225,22 m.

The difference compared to our previous result is about 3 meters. So we have achieved another – even better approximation.

The same process can be done for any number of dividing points. In practice, a dozen of them produce a sufficiently exact value. The new value differs from the one in Figure 4 by one foot (30 cm) only. Because the lift itself must be at least 2 meters high to accommodate passengers, further improvements can hardly give us more realistic answers. The current result is “reasonably good” for “everyday purposes”. Pure mathematicians will probably have difficulties to digest that but in *Teaching Mathematics* we should live with it because mathematics becomes closer to technology in this way. For example, when building constructions like skyscrapers, bridges or lifts, we can only achieve a certain level of precision. We can discuss the limits with our students: *Should it be centimeters or millimeters?* But everyone likely agrees that even if there is a method of calculating the results with a total precision, it is useless for practical purposes because factories and construction companies simply cannot guarantee it.

5. Hurdles and Obstacles

Based on above explanations, one could falsely conclude that all problems of teaching can be solved equally easily. Again, the answer is negative. The main problem lies in the complexity of the change and in the necessity to implement in the present environment. There are many potential opponents: *teachers* (as they know the traditional approaches better), *school administrators* (as they have to invest into re-training, textbooks, and methods of control), *parents* (as they will have smaller chance to help to their kinds and to communicate with them). For these reason we now point to some challenges that require further research. Its results have to make us being capable to explain to general public why the change should be introduced and how – and what we expect as its result.

In this chapter we concentrate on this sort of argumentation.

5.1 Contradictions between Naïve and Formal Solutions

Mathematical methods often cover more than "natural" solutions that can be accomplished using trial-end-error methods. Let us exemplify it:

A shoemaker has been asked to make 100 special-purpose shoes. During the first week he produces nine of them, on the next week eleven, on the third week thirteen. He sees that due to his growing experience and improved skills he will be capable of producing two shoes more during every next week compared to the previous one.

How long will it take for him to produce all pairs?

SHEE	A	B	C	D
1	Week	Shoes	Total	
2	1	9	9	
3	2	11	20	
4	3	13	33	
5	4	15	48	
6	5	17	65	
7	6	19	84	
8	7	21	105	

Figure 9. Calculating the weeks needed for completion of 100 pairs

A naïve solution can exploit a spreadsheet calculation – see Figure 9. In the first column, the weeks are calculated; in the second one, the number of pair produced during the particular week; in the third, the total of pairs. As we see in the cell C8, during the seventh week the desired amount will be achieved.

If we are satisfied with the solution, everything is OK. The problem appears when we would like to demonstrate the general approach based on the sum of the arithmetic progression:

$$S_n = (b_1 + b_n)\frac{n}{2}$$

$$b_n = b_1 + 2(n-1)$$

$$S_n = (2b_1 + 2n - 2)\frac{n}{2} = b_1 n + n^2 - 1 = n^2 + b_1 n - 1$$

$$100 = n^2 + 9n - 1$$

$$n^2 + 9n - 101 = 0$$

The solution of the last expression (of the quadratic equation) is the solution of our original problem using another method. Its roots are 6.5113 and –15.51. The first one corresponds to our approximate (naïve) solution. But what does the second one mean? *Can one hundred pairs of shoes be produced in six and a half weeks as well as in minus fifteen and a half weeks?*

No doubt, none of us would enjoy facing our class asking this question. On the other hand, *Teaching Mathematics* should have prepared "standard answers" for similar situations. In this case, we should claim (with a smile) that we are interested in positive solutions only because "time-travelling is not the subject of our interest". We could even add that we cordially believe that flying with a time-travel-machine one might be capable to produce one hundred pairs of shoes within less than minus sixteen weeks.

Many students will probably be pleased about this sort of humor. If no more, it would relieve the rigid atmosphere – too frequently present in mathematics classes. At the same time, such jokes also introduce students into a sort of "mathematical beauty" based on "what-if-ever". They open door to abstract concepts which are much more difficult to visualize but *may* exist (like irrational or complex numbers).

5.2 Providing Realistic Models

Formulations of problems should use a terminology comprehensible by common public. Whenever possible, they should address situations our students can meet with:

A student body discusses a possibility to organize a fund-raising dinner. The presumed price of a ticket is $24. One member of the organizing committee has found an appropriate space for the event which can be rented for $350. Another one learned about a company providing chairs for $1.50 per night plus free tables. The committee needs to know the minimum number of people which has to come for covering their expenses.

Figure 10. *Revenue*

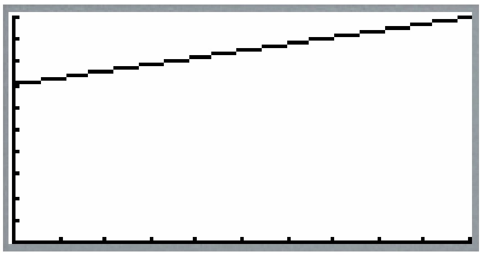

Figure 11. *Costs*

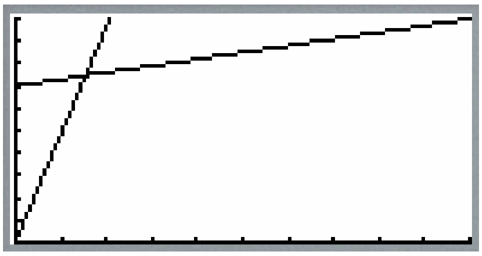

Figure 12. *Break-even point*

Such a problem can be first exploited for training student in reading drawings. Figure 10 shows how the revenue grows with the number of visitors; Figure 11 shows the same for the costs.

There is no reason for hurrying. The teacher should first make certain that all concepts expressed by the drawings are understood well: *Why do both drawing contain straight lines? Why does the first line grow faster? Why the second one does not start at zero?* Only then the final task can be posed: *How do the lines refer to our key task?*

The answers should preferably come from students. Such an approach helps in building their self-confidence in viewing mathematics as “something that can be understood”. Mathematical software allows reading the position of the intersection as [15.55555; 373.33333]. Again, questions should follow: *What is meaning of each number? Do they have a meaning in their present format?* Students easily conclude, that the *x* value must be 16 – the nearest higher integer.

The problem can be easily prolonged. As students are rarely interested in pure business-oriented tasks like break-even points, their interest will be attracted by additional questions as: *The student body plans to buy a TV set to their club for approximately $500. How many people must show up to achieve this level of profit?*

All problems should be preferably solved using several alternative methods. A half of the class can use the above graphic method, the other one the equations:

$$P(x) = R(x) - C(x)$$

$$P(x) = 24x - (1{,}5x + 350) = 22{,}5x - 350$$

$$500 = 22{,}5x - 350$$

$$x = 37{,}77$$

Again we do not use “nice” coefficients producing whole-number-results. We feel them as artificial simplifications appropriate in pre-computer times and used for speeding-up manual calculations. As they are very rare in real-life, the students facing practical problems may feel cheated when asked to interpret non-integer results: “*No one taught as this*.” For we ask: *Why?* Under the presumption of availability of information technology in classrooms makes formulating similar problems more realistic. The students will not only learn to implement them but also become capable of interpreting their results regardles of their „imperfect“ values.

5.3 Introducing Non-linear Models

Our surrounding world is full of non-linear models. Often, we even do not realize their non-linearity e.g. "*Buy two and get one free!*" As we showed with the cosine hyperbolic, introducing non-linear functions does not need to make our teaching material more difficult. Let us discuss a problem similar to the previous one but using a non-linear costs function.

Two girls want to make money to buy Christmas gifts for their relatives and friends. They see their opportunity in making and selling necklaces from glass beans. They realized that first they have to invest $50 to various tools. For each necklace they also need a set of beans. The supplier offers them for the basic price $2, but the price declines by 1 cent per set.

Again, we can ask about the break-even point and profit but we prefer pointing to a different problem. The cost function (Figure 13) is a quadratic function:

$$C(x) = 50 + x(2 - 0.1x) = 50 + 2x - 0.01x^2$$

As its function is a parabola – see Figure 13 – the cost initially grows, then reaches its maximum and starts declining. This is counterintuitive as we all know that with the growing production costs *must* grow. The reason of the irregularity lies in the permanent decrease of the unit price – after certain number of purchased items, the price per unit becomes *negative*. It indicates that similar discount can only be used for a limited number of products. This may lead to searching a better business model – to relative discounts. When the price of the consecutive item is a portion of the previous one (e.g. 90% of it), all prices remain positive. The model is more complex but can easily be calculated for example using a spreadsheet.

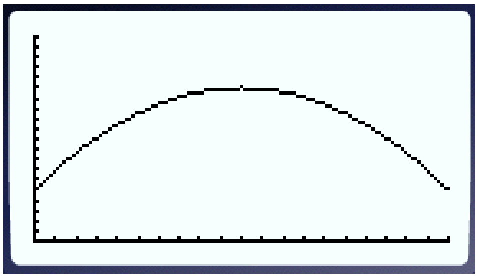

Figure 13. Quadratic costs function

Such an approach may not be appreciated as perfect from the traditionalists' point of view but it is very important for building students' self-confidence. "*I can do it*" should be considered by *Teaching Mathematics* as more important than "*I know all theory behind it*". (How many of us successfully drive cars without knowing the theory of combustion engines?) The most complex models can be omitted – or shifted to *Mathematics for its followers*.

5.4 Syntax, Semantics and Mathematics

Some people feel frustrated by the fact that in mathematics "always works perfect". They may feel happy to learn that it is not always true. One example has been demonstrated by Galileo's error. Another one can be exemplified by problems looking as mathematical ones but wrong otherwise. Compare two problems:

The weight of a horse is 450 kg. What is the weight of 20 horses?

The speed of a running horse is 20 km/hour. What is the speed of 20 horses?

They differ only negligibly from the syntactical point of view. At the same time, there is a huge semantic gap between them. The speed must not be accumulated; the weight can. From the example we see that our considerations must be guided by the problem semantics, not by syntax. Unfortunately, most textbook problems (*A train from A to B …*) are "disinfected" to the manner that does not open questions about their semantics. What about introducing problems like "*A train from Tokyo to San Francisco …*"?

Mathematics is supposed to be about thinking. Rather than fill in all our students' time with calculations and drill, we should submit them material for thinking about mathematical problems, possibly with nonsense interpretations or with interpretations correct under certain conditions and strange in others.:

- *Our old TV set has collapsed so we could not watch any programs. Daddy went to the shop and bought 16 TV's. Mommy also went to the shop and bought 27. How many TV sets do we now have?*
- *A teacher and his class of 23 kids went to the mountains for hiking. After a while, all kids got tired and could not walk. The teacher put them to his backpack and walked carrying them on his back. Later on, eight pupils said that they have rested enough and can walk by themselves. How many children are still in the backpack?*
- *Peter went to the forest behind his house and saw 5 giraffes, 24 gazelles and 7 lions. How many animals did he see?*(The problem might be fair in Kenya but hardly in Slovakia).

Using them, we can teach our students to become capable of recognizing the core of problems, analyze whether it is a mathematical problem or not, finding the "right" method of its solution and become capable of interpreting its result. Even if some people might calculate 400 km/h as the speed of the herd of horses, they should be capable of stopping in a certain moment and start considering: "*Isn't that too much?*" They should learn to interpret the outcomes, assess them critically and ask him/herself: "*Presuming that the result is formally correct – is it also logical?*" The importance of similar questions grows with the application of information technology as we are unable to follow them step by step. IT outcomes are often taken for granted beyond discussion. The previous question combined with common sense can prevent many flaws in applying its deviated results.

6. Conclusion

The results of mathematical education in this present structure are quite sad. Even excellent students do not understand why it is taught. Their picture of a "mathematician" is either "a teacher of Mathematics" or "a researcher solving mathematical problems". They can hardly imagine a mathematician participating in economic, medical or construction problems because they have never been told that some mathematicians do that. Until the public is not convinced about its usefulness, the picture of mathematics and of the role of mathematicians in society will remain distorted. Consequently, we are loosing plenty job opportunities because top managers do not see reasons to invite mathematicians into their teams.

Mathematics is not here just for the pleasure of mathematicians. Its elements were discovered and developed due to practical problems surrounding us. Whilst Latin was a vivid language (at least in the community of highly educated individuals), it was also surviving as a school subject. Our lesson must be straight – *Mathematics Teaching* will continue until people value it as a vivid, vital and no-nonsense subject. Contemporary advanced software tools are much better in performing calculations, formula manipulation and construction of geometric bodies than most people (including highly qualified specialists) are. If Mathematics would only consist of them, its learning would become obsolete and it would follow the destiny of Latin. Thus, we have to:

- Frequently demonstrate that mathematical problems relate to practical life;
- Show alternative methods of solving the same problem and discuss their advantages and disadvantages – see for example [11];
- Stress that finding one practical solution is for users more important than knowing a variety of methods and all theory behind them;
- Train our learners to comprehend the problems, interpret their outcomes and discuss to what degree they are in a good correspondence with our daily experience.

In our paper, we have tried to indicate a few ways of doing so. At the same time, such attempts cannot be isolated. We all should be looking various alternative methods fitting to different "market segments". Attempts have already been made – see for example [12]. What we lack is the feeling of emergency among our community to wake it up to act instantly and thoroughly as a whole.

As a community, we will have to make many experiments to understand which methods work and which not. We should also do "marketing research" that reflect the attitudes of our "customers" and level of their comprehension. A nice and practical method is described in [13]. At the same time, our pupils should feel the same pleasure from our experiments as we do. Mathematics will hardly be an eternal fun – but the proportions might change. Even if it is not always fully understandable, the students should have a feeling that they understand at least a part of it. We humans never understand all processes around us but believe that when we concentrate on some, we can manage them. Give the same feeling to our pupils and students.

We can learn a lot from managerial disciplines how to achieve our aims. I personally hope that the entire community will participate in this learning.

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