Experiencing the Multiple Dimensions of Mathematics with Dynamic 3D Geometry Environments: Illustration with Cabri3D

Colette Laborde
Colette.Laborde@imag.fr
LIG, University Joseph Fourier FRANCE

Abstract: In this paper, we start from the distinction between two processes : iconic and non iconic visualization. Both are involved in solving problems in geometry. The non iconic visualization consists in breaking down an object into parts of same or lower dimension. This cognitive process is critical for solving problems in geometry as very often the reasoning consists in establishing relationships between elements of the figure. However this process is not spontaneous and must be learned. 3D geometry is the source of new problems regarding iconic and non iconic visualization. On the one hand, iconic visualization is not always reliable as it is in 2D geometry, on the other hand non iconic visualization is more complex since it deals with a larger number of kinds of objects, from dimension 0 to dimension 3. The paper examines how 3D dynamic geometry environments with direct manipulation and the tools they offer may enlarge the iconic visualization and assist the non iconic visualization. The example of Cabri3D is used to illustrate the analysis.

1. Multiple dimensions of mathematics

At first glance, the notion of dimension refers to the dimension of geometrical objects such as point (dimension 0), line (dimension1), plane (dimension 2). But we also want to refer in this presentation to external representations or registers (in terms of Duval) used in mathematics for representing mathematical objects and relations or describing them, such as

- graphical representations like diagrams in paper and pencil or on a computer screen
- natural language.

This dual meaning of the notion of dimension is presented in this first section: the geometrical dimension of the objects and the registers in which mathematical objects are expressed or represented.

1.1. Deconstruction of a figure into components

Duval (2005) distinguishes between two ways of “seeing” a figure in 2D geometry or 3D geometry:

- an iconic visualization bearing on the shape: a child recognizes a round shape in a disc or in a circle and is able to distinguish it from a squared shape; the shape of a ball is also easily distinguished from the shape of a cube. The criterion for recognizing the shape bears on the contour of the global object. Shapes must be stable for being recognized.
- a non iconic visualization in which the figure is broken up into components or is transformed into another figure.

The iconic visualization of a cubic box does not consider the faces nor the edges or the vertices of this box. A strong evidence of this is the well-known difficulty children or even older students encounter when counting the number of faces or of edges of a cube even when the cube is available for manipulation. As long as the children do not have structured the cube into for instance the top and bottom faces and lateral faces, they must have recourse to marks on the material cube in order to memorize the faces already counted. This is why enumerating tasks are used in teaching as powerful tasks for fostering the construction of a mental structure for simple solid objects. Similar experiments with other solid objects (pyramids, prisms) carried out with students at the end of high school provide the same evidence (Mariotti 1992, 1996). Constructing a structure for solid objects require identifying the parts of this object of same or lower dimension and their relationships.

For example, a cube can be broken down
- into a set of polyhedrons, like three congruent pyramids with a common vertex and a squared base or two prisms on a triangular base (Fig.1)
- into a set of faces: it can be considered as a prism constructed on a squared face
- into a set of edges : four systems of three edges orthogonal to each other and sharing a common endpoint (points 1, 2, 3 and 4 on Fig.2)
- into a combination mixing edges and faces: it can be structured as made of two parallel squared faces connected by four edges perpendicular to the faces (Fig.3)

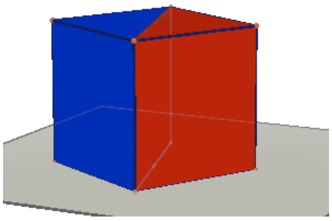

Figure 1 - Two prisms

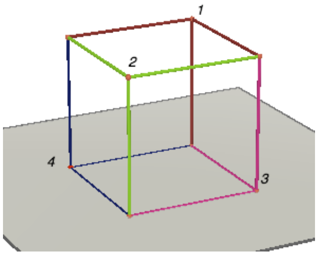

Figure 2 - 4 systems of 3 edges

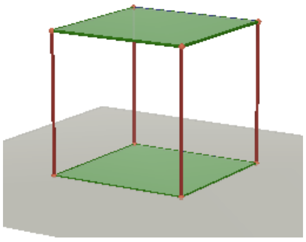

Figure 3 - 2 faces connected by four edges

According to the problem to be solved, one way of breaking down the cube is more appropriate than the other ones. This cognitive process of splitting up an object into subparts of same or lower dimension is the core of the non-iconic visualization and is required in any problem solving in geometry (Duval 2005). This process can be supported by adding some elements on the diagram and/or hiding other elements. The visualization of a cube as made of two prisms is more apparent when the common rectangle of both prisms is drawn.

Although geometry requires both types of visualization, the non iconic visualization is essential for identifying and reasoning about geometrical properties. The non iconic visualization must be learned. This is not an easy task as the iconic visualization which is immediate may sometimes hinder the non iconic visualization. The recognition of a prism is much easier for students when the base is horizontal. The prisms of the deconstruction of the cube (mentioned above) usually are not seen by students on a diagram if their base is not in a horizontal or vertical plane (Fig.4). The iconic visualization when the base is not horizontal or not vertical is more focusing on the corners of the prism than on its parallel edges.

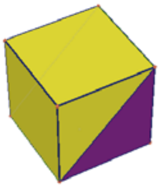

Figure 4 - Prisms in a non prototypical position

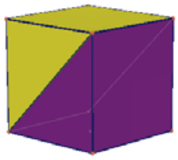

Figure 5 - Prisms in a prototypical position

The possibility of manipulating the cube to move the prisms in a prototypical position helps students see the half cube as a prism (Fig.5). This manipulation allows students to eliminate the conflict between the iconic and the non iconic visualization.

Too often the teaching of mathematics ignores that students have not yet constructed a non iconic visualization and does not help students to be able to develop it. Fishbein (1993) developed the notion of figural concept to give account for this dual role of figural and conceptual in geometry.

1.2. Graphical and textual registers

Each representation of a mathematical object brings some aspects to the fore, whereas it hides other aspects of the same object and thus affects the way the object is conceived. The meaning constructed by the individual is not only affected by the features of the representations available but also by the possible ways to use them. Mathematical activity requires manipulations of and operations on these representations. Various systems of representation in mathematics have been built over time, and these systems affect how we do mathematics. Netz (1999) argues that Greek mathematics was both supported and limited by the available media. Kaput (2001) claims that fundamental representational infrastructures, such as writing systems and algebra, play a major role in determining what and how people think and what they are capable of doing.

Learning mathematics and learning to have a mathematical activity require being able to choose the adequate register for the problem to be solved and possibly to move to another register. What we mean, is that the flexibility of moving between registers is not only supporting the construction of the meaning of a mathematical concept but is essential when «doing» mathematics. Duval (2000, pp.1.63-1.65) claims that understanding a concept requires coordinating at least two registers and being able to move spontaneously and rapidly from one register to another one. In geometry, two registers are indispensable: the graphical register of diagrams and the textual register. As a geometrical figure cannot be entirely determined only from its diagram, a textual description specifying the objects and relationships determining the figure is needed (Parzysz 1988).

It has often been stated that the difficulty of proving in geometry for students lies in the subtle role of diagrams in the elaboration of the proof. On the one hand, diagrams provide ideas about how to justify a statement and it would be impossible to write down the proof of a complex problem without a diagram (Laborde 2005). On the other hand, a proof cannot include elements coming from visual evidence. This subtle role of diagram is far from being used spontaneously by students. The role of the teacher is essential in making students more familiar with this game.

We believe that new 3D geometry environments offer useful tools that can be used by the teacher for the development of both a non-iconic visualization and flexibility between diagram and text. This claim will be illustrated in what follows by means of Cabri3D.

2. 3D dynamic geometry environments with direct manipulation

2.1. Amplifying the reliability of iconic visualization

One of the problems of 3D geometry is that 3D objects can be represented only in 2D even on computer screen unless these objects are represented by material solid objects (such as mock-ups). In 2D geometry, the iconic visualization could hinder the recourse to non iconic visualization but the evidence given by iconic visualization is generally reliable. It is no longer the case in 3D: it is not possible to be sure that two lines intersect from a diagram, or that four points or more are coplanar. It was often observed by teachers that middle school or high school students believe that two lines intersect in 3D because they intersect on the diagram.

The possibility of changing the point of view in 3D dynamic environments with direct manipulation allows the user to obtain immediate visual evidence of such phenomena, as in the following Cabri 3D figures of two non intersecting lines (Fig.6) or of four non planar points (Fig.7).

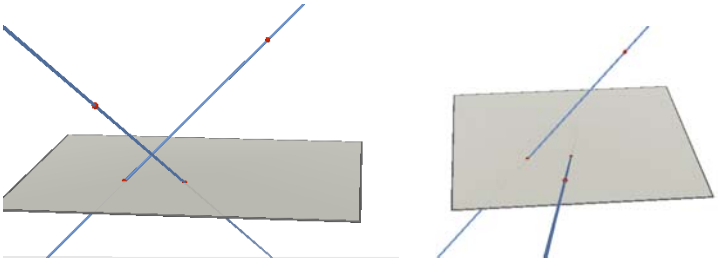

Figure 6 - Two apparently intersecting lines seem no longer intersect after changing the point of view

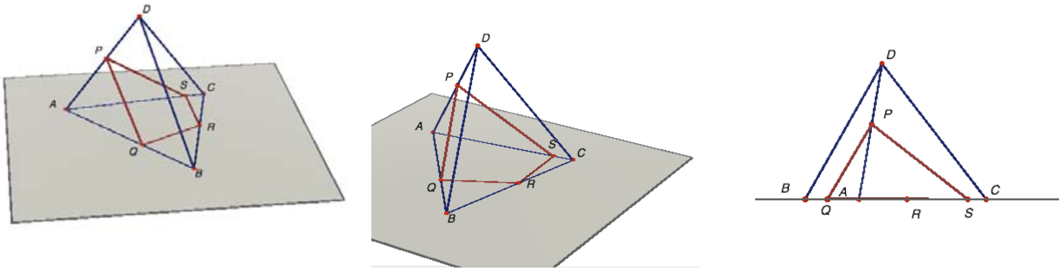

Figure 7 - Four apparently coplanar points seem no longer coplanar after changing the point of view

In 3D dynamic geometry environments, changing the point of view so that three points seem to be on a line provides iconic evidence whether four points are coplanar or not: if the fourth point seems to be outside of the line, the four points are not coplanar.

2.2. Assisting the development of non iconic visualization

Constructing an object by means of Cabri 3D can be done directly in some cases, like for example for usual polyhedra. However in most cases, it can only be done in constructing parts of this object by taking into account their mutual relationships. Construction tasks require a cognitive process of deconstruction of the complex object to construct. The novelty of 3D geometry environments is that this deconstruction is not only possible with 0 or 1 dimensional parts like on paper and pencil but also with 2 or 3 dimensional parts.

Construction tasks of 3D objects in those environments may call thus for an analysis of 3D objects focusing on components of dimension 2 or 3 and so contributes to a better knowledge of space. Before the availability of 3D geometry computer environments, construction by means of 2 or 3 dimensional parts were only possible with mock ups or games.

The very simple example of the cube will be used to illustrate this claim. The teacher asks the students to get rid of the tool « Cube[1]» and to construct a cube from a given square in the base plane. The most spontaneous strategy from students is to construct the edges of the cube and not the faces. They do it by working in the planes of the lateral faces obtained as perpendicular planes containing an edge of the starting square. They construct squares in each plane by using circle. This dimensional deconstruction structures the cube as a net of edges and consists in coming back to

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[1] This possibility of costumizing the toolbar is available in Cabri 3D v.2.1.1

construction of squares in a plane. This is the most usual strategy for students as they are mainly familiar with 2D geometry. The students could use a more economical way of constructing each square by using a 3D tool transferring measurement from a point, i.e. a sphere. They could obtain vertices of the cube as intersection of spheres and perpendicular lines to the base plane (Fig.8). This is sometimes used by students, although this is not their spontaneous strategy as using a sphere to transfer measurement is not possible in paper and pencil.

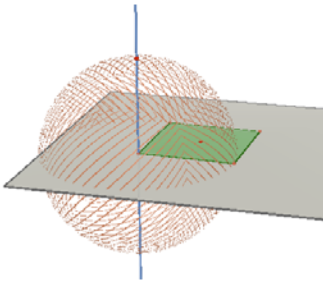

Figure 8 - Construction of a vertex of a cube from the base square

In Cabri3D like environments, there are still other strategies based on a deconstruction of the cube into 2D elements. A lateral square can be considered as the image of the base in a rotation with axis the edge of the base plane (Fig.9). The other lateral faces can be obtained as images of the previous one in rotations around the vertical axis of the cube (Fig10). Those strategies are not used by students. We hypothesize that the construction strategies learned in 2D geometry become obstacles to new strategies specific of 3D.

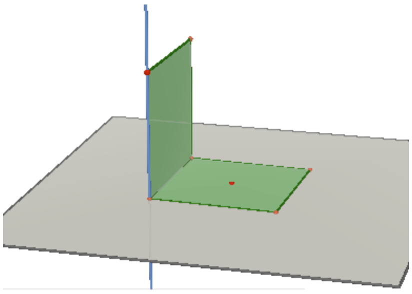

Figure 9 - A lateral face as rotated from the initial square

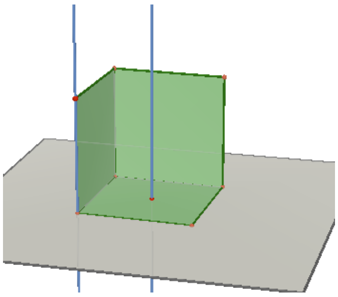

Figure 10 - The second lateral face as rotated from the previous one

We consider these strategies which are made possible by Cabri 3D as interesting from a learning point of view, for two reasons:

- they enlarge the scope of the non iconic vision
- and they use construction tools based on objects and properties of 3D geometry and as such contribute to the learning of 3D geometry.

In a paper and pencil geometry, these objects and properties are not operational for construction tasks, they are only operational in proofs. The strength of Cabri3D like environments is that those objects and properties become operational construction tools. They can be used in action in construction tasks before being used at the level of proof. In construction tasks, students can observe that these strategies provide the expected result. The visual feedback strengthens the power of these properties.

This claim of the contribution of construction strategies based on 2D and 3D objects and properties is convergent with the instrumentation theory. The use of a tool affects the way a subject solve problems depending on the possible actions made possible by the tool. In a first use of a tool, the subject (for instance a student) must learn how to use the tool to solve a certain kind of problems. He must develop an efficient instrumentation of the tool. Doing so the subject constructs knowledge about the tool but also about the knowledge domain of the problem.

Very relevant and beautiful examples of original and efficient instrumentation of Cabri3D are provided by Chuan (2006)[2]. They come from the lecture given by Chuan at ATCM 2006 entitled "Some unmotivated Cabri3D constructions". "Unmotivated" was explained by Chuan as "non algebra, non routine, not found in Euclid, discovered accidentally, tailor made, so short, so beautiful, so fun". For most all these reasons, we consider that facing students with these construction tasks is supporting students learning of a deeper non iconic visualization and thus a better knowledge of space geometry. These constructions are non routine and not found in Euclid because the tools they required were not available. Chuan insists on the efficiency of the constructions (short). This is a critical feature of problems that are able to promote learning of new knowledge according to the theory of didactic situations (Brousseau 1998). A new solving strategy is likely to be constructed by an individual when his/her routine or available strategies are tedious or inoperative for the problem. The beauty of the solution emerges from the conjunction of its efficiency and its unusual character. Let us comment one of the examples given by Chuan: the triangular cupola starting from an equilateral triangle (Fig.11).

Figure 11 – A triangular cupola

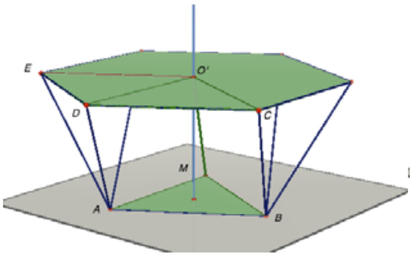

Figure 12 - Axis of the cupola

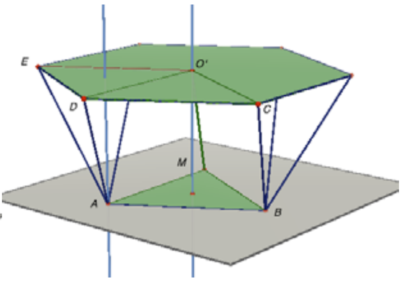

Figure 13 – Tetrahedra OEDA and O'MAB

Constructing the cupola requires analyzing it. The top of the cupola is a regular hexagon whose center O' is on the perpendicular line passing through the center of the given triangle MAB (Fig.12). As soon as the position of O' is determined, the hexagon can be constructed since its side is congruent to AB and the cupola can be constructed as the convex hull of AB and of the hexagonal face (tool Convex Polyhedron). How to obtain the distance of O' to the base plane? One must notice that tetrahedron OEAD is regular and that its altitude is equal to the altitude of tetrahedron O'MAB since they are both regular tetrahedra with congruent edges (Fig.13). Point O' can be constructed as the fourth vertex of the regular tetrahedron constructed on the given triangle MAB.

In this example, a regular tetrahedron is used as measurement transfer tool. This solution is based on a deconstruction of the cupola into 3D and 2D components of the figure. Of course a more traditional deconstruction into 2D and 1D components could be carried out. Such a deconstruction

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[2] at the address sylvester.math.nthu.edu.tw/ d2/talk-atcm2006-unmotivated/

could be used in a strategy based on the determination of the plane ABCD and then on the construction of square ABCD in this plane. This would lead to a longer construction.

2.3 The role of the available tools in the environment

The available tools of the environment affect very much the possible construction strategies. We have just seen how the availability of regular polyhedra from a regular polygon makes possible the transfer of measures. Some other tools may play an efficient role in construction tasks avoiding long construction processes.

This is the case of the tool providing the convex polyhedron hull of four points or more (in particular of two polygons, of a segment and a polygon…). Constructing a convex polyhedron requires thus identifying the minimal number of elements of lower dimension determining the polyhedron.

The sphere is also a tool offering a transfer of measure from a point in any direction. The transfer of a measure in a plane from a point is also possible by using a circle around an axis (Fig. 14). Transferring the length of a segment on a perpendicular line from one of its endpoint can be done in the same way by using a square (Fig.15). The transfer of angles of some regular polygons is also possible by using the tools regular polygons around a line or a segment (Fig.16). All these tools are based on geometric properties of regular polygons and of circles.

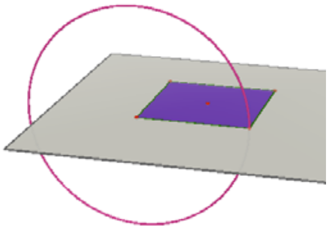


Figure 14 – Measurement transfer by means of a circle

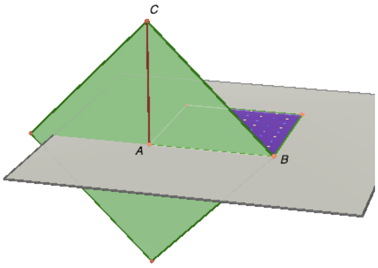


Figure 15 – a square

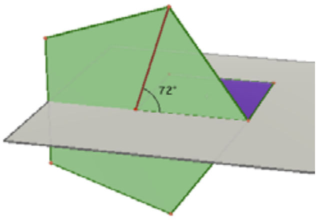


Figure 16- Transfer of an angle

The non iconic visualization is called twice

- in identifying that on the 3D figure to be constructed, two segments (elements of dimension 1) are congruent
- in adding another object of dimension 2 or 3 in which two segments with the same relative position are also congruent.

A process of going down and up in the dimensions of the objects is very much involved in the use of these tools. Therefore we consider that construction tasks in 3D geometry environments offering this kind of tools is very demanding in terms of geometric knowledge and conversely can be used by teachers to promote the development of non iconic visualization in 3D.

Transformations offered by Cabri 3D are also tools that can be used to construct 3D complex objects. Identifying that one part of the object is the image of another part in a transformation is also a matter of non iconic visualization.

The interface representing continuously the image in the construction process (Fig.17) of a rotation around an axis when the angle is increased continuously from 0 until its target value provides iconic visualization simulating a real motion in space and hence supports the non iconic visualization. We assume that this possibility of seeing a continuous movement between an object and its image can be used by the teacher to provide imagery of movements in space that very often students do not have and to relate these movements to geometrical transformations. Students can be asked to simulate a triangle rotating around one of its sides, or a face of a polyhedron to rotate around one of its edges, or a polyhedron around an axis.

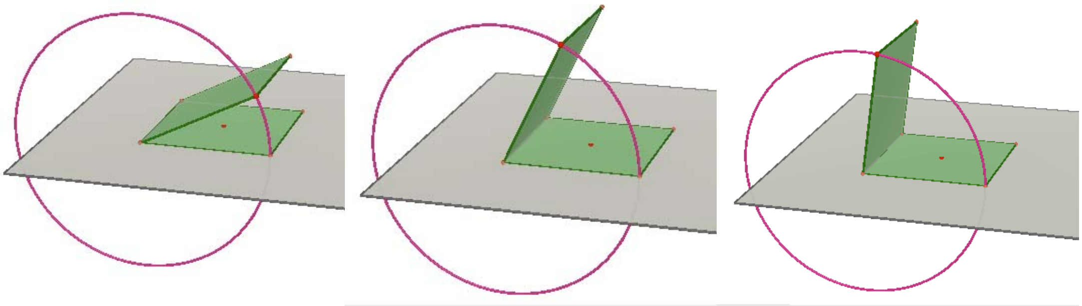
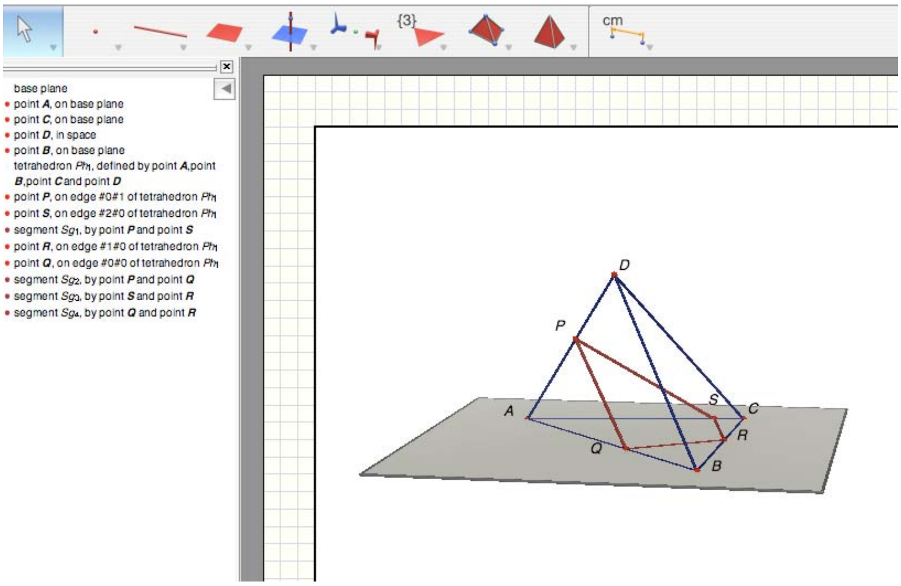


Figure 17 – A rotating square around an edge of another square

2.4 Complementary roles of graphical and textual registers

Let us come back to the example of four points apparently coplanar. As said above, changing the point of view allows the students to augment their iconic visualization and to invalidate this fact. However the reason why four points are coplanar or non coplanar can only be found by using theoretical knowledge. The textual description of a figure offered by Cabri3D provides the objects and their relationships (geometrical properties) that define this figure.



base plane
- point **A**, on base plane
- point **C**, on base plane
- point **D**, in space
- point **B**, on base plane

tetrahedron Ph_1, defined by point **A**,point **B**,point **C** and point **D**
- point **P**, on edge #0#1 of tetrahedron Ph_1
- point **S**, on edge #2#0 of tetrahedron Ph_1
- segment Sg_1, by point **P** and point **S**
- point **R**, on edge #1#0 of tetrahedron Ph_1
- point **Q**, on edge #0#0 of tetrahedron Ph_1
- segment Sg_2, by point **P** and point **Q**
- segment Sg_3, by point **S** and point **R**
- segment Sg_4, by point **Q** and point **R**

The surprise of the students discovering that four points they expected to be coplanar are not can be used by the teacher to motivate students to prove why. Students often have difficulties in not using evidence given by the figure in their proof: it is clear that three points are in the same plane and the fourth one is not. The existence of the textual description can be utilized by the teacher: the proof of the fact that these four points are not coplanar must be based only on information items given by the textual description.

The teacher can ask: Why do we know from the description that R, Q and S are in the same plane? Why do we know from the description that P is not in this plane? The link between the textual description offered by the environment can help students to find in the description all information items about an object. When clicking on the object in the diagram, all the occurrences of the object in the textual description are highlighted and conversely when clicking on an object of the description, all the other occurrences of the same object are highlighted and its representation in the diagram is flashing. While the description provides an objective criterion for insuring that the proof is only based on properties used to build the figure, the interactive link between text and diagram allows the student to reason by using non iconic visualization coming from the diagram. It offers a way of overcoming the paradoxical situation which students face: they must elaborate a proof with the help of the diagram but are not allowed to refer to the diagram in the text of the proof.

Such a proof requires to make explicit some theorems and axioms of 3D geometry. This can be used to make university students, in particular preservice teachers, aware of the axiomatic system of geometry. Cabri II on the TI 92 was already used in this way to introduce university students to the axiomatic system of geometry and to do formal proofs (Perry Carrasco et al. 2006) or Italian high school students to construct a system of axioms (Mariotti 2000).

Teachers often consider that 3D geometry is hard matter to learn. In this paper, we pointed out two cognitive processes contributing to the difficulty of 3D geometry: iconic visualization and non iconic visualization. These processes are an essential part of any geometrical activity. We attempted to show that tools available in new 3D dynamic geometry environments may not only assist these cognitive processes but enlarge their range. Since it makes accessible in particular operations on 2D and 3D objects, it may extend non iconic visualization to those objects. Of course the teacher is still needed. One of his/her roles is to design challenging tasks requiring an extended non iconic visualization. Such tasks can even be fun when, for example, they consist in reproducing dynamic 3D objects given on the screen of the computer as in Chuan's examples.

References

Chuan, J. (2006) Some unmotivated Cabri 3D constructions, Lecture at ATCM 2006, Available: http://sylvester.math.nthu.edu.tw/d2/talk-atcm2006-unmotivated/ (Accessible September 20, 2007).

Duval, R. (2000) Basic issues for research in mathematics education, In T. Nakahara, M. Koyama (Eds.), *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education*, (Vol 1, pp. 55-69) Hiroshima: Hiroshima University.

Duval, R. (2005) Les conditions cognitives de l'apprentissage de la géométrie : développement de la visualisation, différenciation des raisonnements et coordination de leurs fonctionnements, In: *Annales de didactique et sciences cognitives*, 5-53, Strasbourg, France: IREM, Université Louis Pasteur.

Fishbein, E. (1993) The theory of figural concepts. *Educational Studies in Mathematics, 24*(2), 139-162.

Kaput, J. (2001, January). *Changing representational infrastructures changes most everything: The case of SimCalc algebra, and calculus*. Paper presented at the NAS symposium on improving learning with informational technology. Washington, DC. Available: http://www.simcalc.umassd.edu/library/library_2001.php. (Accessible September 20, 2007).

Laborde C. (2005) The hidden role of diagrams in students' construction of meaning in geometry In J. Kilpatrick, C. Hoyles and O. Skovsmose (Eds.), *Meaning in mathematics education* (pp.1–21). (pp.159-180) Dordrecht : Kluwer Academic Publishers

Mariotti A. (1992) Le raisonnement géométrique : une dialectique entre les aspects figural et conceptuel, *Topologie structurale*, 18, 9-17.

Mariotti, A. (1996) *The interaction between images and concepts in geometrical reasoning* PhD dissertation of the Tel Aviv University.

Mariotti, A. (2000) Introduction to proof : the mediation of a dynamic software environment. *Educational Studies in Mathematics, 44 (1-3)*, 25–53.

Netz, R. (1999). *The shaping of deduction in Greek mathematics*. Cambridge, UK: Cambridge University Press.

Parzysz, B. (1988) Knowing vs seeing: Problems of the plane representation of space geometry figures. Educational Studies in Mathematics 19(1), 79-92.

Perry Carrasco, P. Camargo Uribe, L., Samper, C. Rojas Morales, C. (2006) *Actividad demostrativa en la formacion inicial de profesores de matematicas*. Bogotá: Universidad Pedagógica Nacional, 2006. 363 p.; incl. ref. 2006 Impreso Es.