# Solving Harder Problems with Lesser Mathematics*

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There are many computer software packages for solving/proving difficult mathematical problems/theorems in automatic mode, but most of them are implemented as a black-box manipulation. It will be illustrated in this talk how to make use of computer programs to deal with some interesting and harder problems in a "readable" mode or a computer-aided mode, which can be more accessible by more audience.

## 1 Introduction

Let us begin with a more simple example. Given a homogeneous polynomial in $x, y, z$,

$$P(x,y,z) = 3\,x^3 - 3\,x^2y - 3\,x^2z - 3\,xy^2 + 9\,xyz - 3\,xz^2 + 8\,y^3 - 8\,y^2z - 8\,yz^2 + 8\,z^3,$$

show $P(x,y,z) \geq 0$ provided $x, y, z$ all are nonnegative.

There are various methods to verify the nonnegativity of $P$, but I prefer a certain one which uses a little mathematics. That is, *split the quantities $x, y$ and $z$ into smaller nonnegative ones, then, collect the terms and see whether the coefficients all are nonnegative.*

One may use the following linear transformation for splitting:

$$\begin{aligned} x &= t_1 + t_2 + t_3 \\ y &= t_2 + t_3 \\ z &= t_3 \end{aligned}$$

that is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} \tag{1}$$

where $t_1 \geq 0, t_2 \geq 0, t_3 \geq 0$ if and only if $x \geq y \geq z$.

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By (1), $P(x,y,z)$ transforms into a polynomial in $t_1,t_2,t_3$:

$$P_1(t_1,t_2,t_3)=3\,t_1^3+6\,t_1^2t_2+3\,t_1^2t_3+3\,t_1t_2t_3+5\,t_2^3+13\,t_2^2t_3. \tag{2}$$

All the coefficients of (2) are nonnegative, so $P(x,y,z)=P_1(t_1,t_2,t_3)\geq 0$ when $x\geq y\geq z$.

Otherwise, when $x\geq z\geq y$, we use the following transformation in stead of (1),

$$\begin{bmatrix} x \\ z \\ y \end{bmatrix}=\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} \tag{3}$$

that keeps $t_1\geq 0, t_2\geq 0, t_3\geq 0$, and $P(x,y,z)$ is also transformed into

$$P_1(t_1,t_2,t_3)=3\,t_1^3+6\,t_1^2t_2+3\,t_1^2t_3+3\,t_1t_2t_3+5\,t_2^3+13\,t_2^2t_3,$$

so $P(x,y,z)=P_1(t_1,t_2,t_3)\geq 0$ for this instance.

Analogously, when $y\geq x\geq z$ or $z\geq x\geq y$, we use the following transformations respectively,

$$\begin{bmatrix} y \\ x \\ z \end{bmatrix}=\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}, \quad \begin{bmatrix} z \\ x \\ y \end{bmatrix}=\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$$

that both keep $t_1\geq 0, t_2\geq 0, t_3\geq 0$, respectively, and $P(x,y,z)$ is transformed into

$$P_2(t_1,t_2,t_3)=8\,t_1^3+21\,t_1^2t_2+13\,t_1^2t_3+15\,t_1t_2^2+23\,t_1t_2t_3+5\,t_2^3+13\,t_2^2t_3, \tag{4}$$

so $P(x,y,z)=P_2(t_1,t_2,t_3)\geq 0$ for the two instances.

Finally, when $y\geq z\geq x$ or $z\geq y\geq x$, we use the following transformations respectively,

$$\begin{bmatrix} y \\ z \\ x \end{bmatrix}=\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}, \quad \begin{bmatrix} z \\ y \\ x \end{bmatrix}=\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$$

that both keep $t_1\geq 0, t_2\geq 0, t_3\geq 0$, respectively, and $P(x,y,z)$ is transformed into

$$P_3(t_1,t_2,t_3)=8\,t_1^3+16\,t_1^2t_2+13\,t_1^2t_3+3\,t_1t_2t_3+3\,t_2^2t_3, \tag{5}$$

so $P(x,y,z)=P_3(t_1,t_2,t_3)\geq 0$ for the last two instances.

Let us sum up the procedure above: There are 6 different orders of $x,y,z$ sorting by size. Each of the orders corresponds a linear transformation which splits $x,y,z$ into smaller nonnegative quantities $t_1,t_2,t_3$ and transforms $P(x,y,z)$ into a polynomial in $t_1,t_2,t_3$. Since every resulting polynomial is of nonnegative coefficients, the original polynomial $P$ is nonnegative as well.

This proof uses a little mathematics only: the sum and the product of nonnegative numbers must be nonnegative, too. The trick like this has been frequently applied to symmetric polynomials by so many people for so many years that I cannot list all the names and jobs. What we will try here is to apply the "splitting" method to nonnegativity decision for much more polynomials including asymmetric and symmetric ones which appear in various problems.

## 2 Difference Substitution

Referring to the example in Section 1, we call the set $\{P_1, P_2, P_3\}$ the *Difference Substitution of P* and denoted by $\mathrm{DS}\,(P)$, because

$$\begin{cases} x = t_1 + t_2 + t_3 \\ y = t_2 + t_3 \\ z = t_3 \end{cases} \quad \text{means} \quad \begin{cases} t_1 = x - y \\ t_2 = y - z \\ t_3 = z, \end{cases} \tag{6}$$

i.e., $\{t_1, t_2, t_3\}$ is the difference sequence of $\{x, y, z\}$, and

$$\begin{cases} x = t_1 + t_2 + t_3 \\ z = t_2 + t_3 \\ y = t_3 \end{cases} \quad \text{means} \quad \begin{cases} t_1 = x - z \\ t_2 = z - y \\ t_3 = y, \end{cases} \tag{7}$$

i.e., $\{t_1, t_2, t_3\}$ is the difference sequence of $\{x, z, y\}$, etc.

Generally speaking, the Difference Substitution of a ternary polynomial has up to 6 members. Let us see another example:

$$Q(x, y, z) = 2\,x^4 - 3\,x^2y^2 - 6\,x^2yz + 9\,x^2z^2 + 2\,xy^3 - 6\,xyz^2 - 4\,xz^3 + 2\,y^3z + 3\,y^2\,z^2 + z^4,$$

and then compute its Difference Substitution, $\mathrm{DS}\,(Q) = \{Q_1, Q_2, Q_3, Q_4, Q_5, Q_6\}$, where

$$\begin{aligned}
Q_1 &= 2\,t_1^4 + 8\,t_1^3t_2 + 8\,t_1^3t_3 + 9\,t_1^2t_2^2 + 12\,t_1^2t_2t_3 + 12\,t_1^2t_3^2 + 4\,t_1t_2^3 + t_2^4,\\
Q_2 &= 2\,t_1^4 + 8\,t_1^3t_2 + 8\,t_1^3t_3 + 21\,t_1^2t_2^2 + 36\,t_1^2t_2t_3 + 12\,t_1^2t_3^2 + 22\,t_1t_2^3 + 48\,t_1t_2^2t_3 + 24\,t_1t_2t_3^2\\
&\quad + 8\,t_2^4 + 20\,t_2^3t_3 + 12\,t_2^2t_3^2,\\
Q_3 &= 2\,t_1^3t_2 + 4\,t_1^3t_3 + 3\,t_1^2t_2^2 + 12\,t_1^2t_2t_3 + 12\,t_1^2t_3^2 + t_2^4,\\
Q_4 &= 2\,t_1^3t_2 + 4\,t_1^3t_3 + 9\,t_1^2t_2^2 + 24\,t_1^2t_2t_3 + 12\,t_1^2t_3^2 + 12\,t_1t_2^3 + 36\,t_1t_2^2t_3 + 24\,t_1t_2t_3^2 + 6\,t_2^4\\
&\quad + 16\,t_2^3t_3 + 12\,t_2^2t_3^2,\\
Q_5 &= t_1^4 + 3\,t_1^2t_2^2 + 10\,t_1t_2^3 + 12\,t_1t_2^2t_3 + 8\,t_2^4 + 20\,t_2^3t_3 + 12\,t_2^2t_3^2,\\
Q_6 &= t_1^4 + 4\,t_1^3t_2 + 9\,t_1^2t_2^2 + 12\,t_1t_2^3 + 12\,t_1t_2^2t_3 + 6\,t_2^4 + 16\,t_2^3t_3 + 12\,t_2^2t_3^2.
\end{aligned}$$

Noting all coefficients of the polynomials in $\mathrm{DS}\,(Q)$ are nonnegative, by the same argument as that for $P(x, y, z)$, we conclude that $Q(x, y, z) \geq 0$ whenever $x, y, z$ all are nonnegative.

In general, for the $n$-variant polynomials, there are $n!$ different orders of $x_1, x_2, \cdots, x_n$ sorting by size. To each order, say, $x_1 \geq x_2 \geq \cdots \geq x_n$, corresponds a "splitting" transformation,

$$\begin{cases} x_1 = t_1 + t_2 + \cdots + t_n \\ x_2 = t_2 + \cdots + t_n \\ \quad \cdots \\ x_n = t_n, \end{cases} \tag{8}$$

and $t_1, t_2, \cdots, t_n$ is just the difference sequence of $x_1, x_2, \cdots, x_n$.

Analogously, the Difference Substitution of an $n$-variant polynomial $F(x_1, x_2, \cdots, x_n)$ has up to $n!$ members. If all the coefficients of these members are nonnegative, then $F \geq 0$ whenever $x_1, x_2, \cdots, x_n$ all are nonnegative, in other words, $F$ is positive semi-definite on $\mathbf{R}_+^n$.

For brevity, a set of polynomials is called *trivially nonnegative* if all the coefficients of its members are nonnegative. It must be pointed out that DS$(F)$ to be trivially nonnegative is only a sufficient condition for $F$ to be positive semi-definite on $\mathbf{R}_+^n$, but not a necessary one. In the practice of proving inequalities, however, we often meet polynomials whose Difference Substitutions are trivially nonnegative, so that these inequalities are proven very easily.

`Problem 1.` Show that the following polynomial is positive semi-definite on $\mathbf{R}_+^3$,

$$F = x^3 + y^3 + z^3 - x^2y - xy^2 - x^2z - xz^2 - y^2z - yz^2 + 3\,xyz. \tag{9}$$

This originated from the well-known Robinson's polynomial [3]. Here DS$(F)$ consists of a single polynomial, $t_1^3 + 2\,t_1^2t_2 + t_1^2t_3 + t_1t_2t_3 + t_2^2t_3$, so it is trivially nonnegative, hence $F$ positive semi-definite on $\mathbf{R}_+^3$.

`Problem 2.` Show that the following inequality holds on $\mathbf{R}_+^3$,

$$\left(\frac{1}{2}(x^2+y^2+z^2)(x+y+z) - xyz\right)^2 \le \frac{1}{2}\left(x^2+y^2+z^2\right)^3. \tag{10}$$

In other words, show the polynomial

$$\begin{aligned}F = {} & x^6 - 2\,x^5y - 2\,x^5z + 3\,x^4y^2 + 2\,x^4yz + 3\,x^4z^2 - 4\,x^3y^3 - 4\,x^3z^3 \\ & + 3\,x^2y^4 + 2\,x^2y^2z^2 + 3\,x^2z^4 - 2\,xy^5 + 2\,xy^4z + 2\,xyz^4 - 2\,xz^5 + y^6 \\ & - 2\,y^5z + 3\,y^4z^2 - 4\,y^3z^3 + 3\,y^2z^4 - 2\,yz^5 + z^6\end{aligned}$$

to be positive semi-definite on $\mathbf{R}_+^3$. ( `http://www.mathlinks.ro/Forum/topic-54136.html`). Here DS$(F)$ consists of a single polynomial,

$$\begin{aligned}& t_1^6 + 4\,t_1^5t_2 + 2\,t_1^5t_3 + 8\,t_1^4t_2^2 + 8\,t_1^4t_2t_3 + 3\,t_1^4t_3^2 + 8\,t_1^3t_2^3 + 12\,t_1^3t_2^2t_3 \\ & + 12\,t_1^3t_2t_3^2 + 4\,t_1^3t_3^3 + 4\,t_1^2t_2^4 + 8\,t_1^2t_2^3t_3 + 20t_1^2t_2^2t_3^2 + 20\,t_1^2t_2t_3^3 + 7\,t_1^2t_3^4 + 16\,t_1t_2^3t_3^2 \\ & + 36\,t_1t_2^2t_3^3 + 32\,t_1t_2t_3^4 + 10\,t_1t_3^5 + 8\,t_2^4t_3^2 + 24\,t_2^3t_3^3 + 32\,t_2^2t_3^4 + 20\,t_2t_3^5 + 5\,t_3^6,\end{aligned}$$

so it is trivially nonnegative, hence $F$ positive semi-definite on $\mathbf{R}_+^3$.

`Problem 3.` Show that the following polynomial is positive semi-definite on $\mathbf{R}_+^4$,

$$F = a^4b + b^4c + c^4d + d^4a - abcd(a+b+c+d). \tag{11}$$

There are 24 different orders of $a, b, c, d$ sorting by size. Each of the orders corresponds a linear transformation which splits $a, b, c, d$ into smaller nonnegative quantities. We implemented all the 24 transformations by computer and obtained DS$(F)$ which consists of 6 polynomials with nonnegative coefficients, so it is trivially nonnegative, hence $F$ positive semi-definite on $\mathbf{R}_+^4$. This problem comes from `http://www.mathlinks.ro/Forum/topic-45218.html`

`Problem 4.` Show that the following polynomial is positive semi-definite on $\mathbf{R}_+^5$,

$$\begin{aligned}F = {} & 1056\,x_4x_5^4 + 744x_4^4x_5 + 1120x_3x_5^4 + (672\,x_2 + 192\,x_5 + 352\,x_4 + 512\,x_3)x_1^4 \\ & + (-3360x_5x_4 + 912\,x_5^2 - 1440x_2x_3 + 752\,x_3^2 + 672\,x_2^2 - 2400x_3x_4 - 2400x_5x_2 \\ & + 832\,x_4^2 - 2880x_5x_3 - 1920x_4x_2)x_1^3 + 1224x_3^4x_4 + 1064x_5x_3^4 + (320x_4^3 + 2016x_2^2x_3 \\ & - 96x_2^3 - 3456x_3x_5x_4 + 528x_5^3 + 3312x_5^2x_4 + 112x_3^3 + 2736x_3^2x_4 + 2016x_2^2x_5\end{aligned}$$

$$
\begin{aligned}
&+ 3312x_5^2x_2 + 3312x_5^2x_3 + 2736x_3^2x_5 + 2592x_2x_4^2 - 3456x_2x_5x_4 + 1872x_2x_3^2\\
&+ 2016x_4x_2^2 - 3456x_2x_3x_4 + 3456x_4^2x_5 - 3456x_3x_5x_2 + 2592x_3x_4^2)x_1^2 + 1200x_2^4x_3\\
&+ (2736x_2^2x_3^2 - 4992x_5^3x_2 - 3744x_2^3x_5 - 4800x_4^3x_2 - 2784x_2^3x_3 - 4992x_3x_4^3\\
&- 3264x_4x_2^3 + 3456x_4^2x_2^2 - 4320x_3^3x_4 + 1152\,x_2^2x_3x_4 + 2304x_2x_3^2x_4 + 1152\,x_2^2x_4x_5\\
&+ 2304x_3x_4^2x_5 + 2304x_2x_3^2x_5 + 1152\,x_2^2x_3x_5 + 2304x_2x_4^2x_5 + 1152\,x_3^2x_4x_5\\
&- 4608x_3^3x_2 + 336x_2^4 + 1248x_5^4 + 1448x_4^4 + 1144x_3^4 + 4752x_5^2x_4^2 + 3744x_4^2x_3^2\\
&- 5184x_5^3x_3 + 4176x_5^2x_2^2 - 5376x_4x_5^3 - 4800x_5x_3^3 + 4464x_5^2x_3^2 - 5856x_4^3x_5)x_1\\
&+ 1184x_5^4x_2 + 528x_2^3x_3^2 + 384x_3^2x_4^3 - 4992x_3^3x_2x_5 + 384x_3^3x_4^2 + 240x_5^2x_4^3 + 1320x_4^4x_3\\
&+ 144x_3^3x_2^2 + 1080x_3^4x_2 + 432x_1^5 + 560x_5^3x_2^2 + 880x_2^4x_5 + 688x_2^3x_5^2 + 1152x_5^5\\
&- 5376x_5^3x_2x_3 - 5568x_5^3x_2x_4 + 3600x_5^2x_2^2x_3 + 3024x_2^2x_3^2x_5 - 5280x_4x_2^3x_5\\
&+ 3744x_4^2x_2^2x_5 + 3024x_3^2x_4x_2^2 + 3744x_3^2x_4^2x_2 - 5184x_3x_4^3x_2 + 2880x_3x_4^2x_2^2\\
&- 4512x_3^3x_4x_2 - 4320x_2^3x_3x_4 + 3600x_2^2x_4x_5^2 + 3888x_3^2x_4x_5^2 + 4752x_2x_4^2x_5^2\\
&+ 4464x_2x_3^2x_5^2 - 6240x_3x_4^3x_5 - 5760x_3x_4x_5^3 + 4752x_3x_5^2x_4^2 + 4032x_3^2x_4^2x_5\\
&- 7200x_3^3x_4x_5 - 6048x_4^3x_2x_5 - 4800x_2^3x_3x_5 + 864x_2^5 + 1224x_4^5 + 1128x_3^5\\
&+ 608x_4^2x_2^3 + 352\,x_4^3x_2^2 + 1384x_4^4x_2 + 1040x_2^4x_4 + 624x_4^2x_5^3 + 464x_5^2x_3^3\\
&+ 592\,x_5^3x_3^2 - 3456x_3x_2^2x_4x_5 + 1152\,x_2x_3^2x_4x_5 + 2304x_3x_2x_4^2x_5.
\end{aligned}
$$

There are 120 different orders of $x_1, x_2, x_3, x_4, x_5$ sorting by size. Each of the orders corresponds a linear transformation which splits $x_1, x_2, x_3, x_4, x_5$ into smaller nonnegative quantities. We implemented all the 120 transformations by computer and obtained DS $(F)$ which consists of 120 polynomials with nonnegative coefficients, so it is trivially nonnegative, hence $F$ positive semi-definite on $\mathbf{R}_+^5$. This problem comes from " `http://guestbook.nease.net/read.php?user=zgbdsyjxz&id=1118121244&curpage=36&page=2` ".

`Problem 5.` Show that the following inequality holds on $\mathbf{R}_+^5$,

$$
\frac{a_1}{a_2+a_3} + \frac{a_2}{a_3+a_4} + \frac{a_3}{a_4+a_5} + \frac{a_4}{a_5+a_1} + \frac{a_5}{a_1+a_2} \geq \frac{5}{2}. \tag{12}
$$

In other words, show the following polynomial to be positive semi-definite on $\mathbf{R}_+^5$,

$$
\begin{aligned}
F = {} & 2\,a_1^3a_3a_4 + 2\,a_1^3a_3a_5 + 2\,a_1^3a_4^2 + 2\,a_1^3a_4a_5 + 2\,a_1^2a_2^2a_4 + 2\,a_1^2a_2^2a_5 + 2\,a_1^2a_2a_3^2 + a_1^2a_2a_3a_4\\
&- a_1^2a_2a_3a_5 - 3a_1^2a_2a_4^2 - 3a_1^2a_2a_4a_5 + 2\,a_1^2a_3^3 - 3a_1^2a_3^2a_4 - 5a_1^2a_3^2a_5 - 5a_1^2a_3a_4^2\\
&- 3a_1^2a_3a_4a_5 + 2\,a_1^2a_3a_5^2 + 2\,a_1^2a_4^2a_5 + 2\,a_1^2a_4a_5^2 + 2\,a_1a_2^3a_4 + 2\,a_1a_2^3a_5 + 2\,a_1a_2^2a_3^2\\
&- a_1a_2^2a_3a_4 - 3a_1a_2^2a_3a_5 - 5a_1a_2^2a_4^2 - 3a_1a_2^2a_4a_5 + 2\,a_1a_2^2a_5^2 + 2\,a_1a_2a_3^3 - 3a_1a_2a_3^2a_4\\
&- 3a_1a_2a_3^2a_5 - 3a_1a_2a_3a_4^2 + a_1a_2a_3a_5^2 + 2\,a_1a_2a_4^3 + a_1a_2a_4^2a_5 - a_1a_2a_4a_5^2 + 2\,a_1a_3^3a_5\\
&+ 2\,a_1a_3^2a_4^2 + a_1a_3^2a_4a_5 - 3a_1a_3^2a_5^2 + 2\,a_1a_3a_4^3 - a_1a_3a_4^2a_5 - 3a_1a_3a_4a_5^2 + 2\,a_2^3a_4a_5\\
&+ 2\,a_2^3a_5^2 + 2\,a_2^2a_3^2a_5 + 2\,a_2^2a_3a_4^2 + a_2^2a_3a_4a_5 - 3a_2^2a_3a_5^2 + 2\,a_2^2a_4^3 - 3a_2^2a_4^2a_5 - 5a_2^2a_4a_5^2\\
&+ 2\,a_2a_3^3a_5 + 2\,a_2a_3^2a_4^2 - a_2a_3^2a_4a_5 - 5a_2a_3^2a_5^2 + 2\,a_2a_3a_4^3 - 3a_2a_3a_4^2a_5 - 3a_2a_3a_4a_5^2\\
&+ 2\,a_2a_3a_5^3 + 2\,a_2a_4^2a_5^2 + 2\,a_2a_4a_5^3 + 2\,a_3^2a_4a_5^2 + 2\,a_3^2a_5^3 + 2\,a_3a_4^2a_5^2 + 2\,a_3a_4a_5^3.
\end{aligned}
$$

There are 120 different orders of $a_1, a_2, a_3, a_4, a_5$ sorting by size. Each of the orders corresponds a linear transformation which splits $a_1, a_2, a_3, a_4, a_5$ into smaller nonnegative quantities. We implemented all the 120 transformations by computer and obtained DS$(F)$ which consists of 24 polynomials with nonnegative coefficients, so it is trivially nonnegative, hence $F$ positive semi-definite on $\mathbf{R}_+^5$. This is the so-called " 5-cyclic inequality".

`Problem 6.` Show that the following inequality holds on $\mathbf{R}_+^{10}$,

$$F = \sum_{k=1}^{10} a_k^{10} - 10 \prod_{k=1}^{10} a_k \geq 0.$$

Here DS$(F)$ consists of a single polynomial with nonnegative coefficients, hence $F$ positive semi-definite on $\mathbf{R}_+^{10}$. The computation was implemented by computer.

It is obvious for any *symmetric* polynomial that the Difference Substitution consists of a single member only, because all the distinct orders of the variables lead to the same polynomial.

`Problem 7.` Show that the following polynomial is positive semi-definite on $\mathbf{R}_+^4$,

$$\begin{aligned} F = {} & (-x_3^2 - 2\,x_4x_1 + 6x_1^2 + 6x_2^2 + 4x_2x_1 - x_4^2 - 2\,x_2x_3 - 2\,x_3x_1 - 2\,x_4x_2)(x_1 - x_2)^{1000} \\ & + (-2\,x_4x_1 - 2\,x_3x_1 - 2\,x_4x_2 + 4x_3x_4 + 6x_4^2 - x_1^2 - 2\,x_2x_3 - x_2^2 + 6x_3^2)(x_3 - x_4)^{1000} \\ & + (6x_4^2 - x_3^2 - 2\,x_2x_3 + 6x_2^2 - 2\,x_2x_1 - x_1^2 - 2\,x_3x_4 + 4x_4x_2 - 2\,x_4x_1)(x_2 - x_4)^{1000} \\ & + (6x_3^2 - x_1^2 - x_4^2 + 6x_2^2 - 2\,x_3x_1 - 2\,x_2x_1 - 2\,x_3x_4 - 2\,x_4x_2 + 4x_2x_3)(x_2 - x_3)^{1000} \\ & + (-x_2^2 + 6x_1^2 - 2\,x_3x_1 - 2\,x_4x_2 + 6x_4^2 - 2\,x_3x_4 + 4x_4x_1 - x_3^2 - 2\,x_2x_1)(x_4 - x_1)^{1000} \\ & + (-2\,x_4x_1 - 2\,x_2x_1 - 2\,x_3x_4 - x_4^2 + 6x_3^2 - 2\,x_2x_3 + 6x_1^2 + 4x_3x_1 - x_2^2)(x_3 - x_1)^{1000}. \end{aligned}$$

This quaternary polynomial is of degree 1002 ! Here DS$(F)$ consists, however, of one member only because $F$ is symmetric. And the coefficients of the single member are all nonnegative, hence $F$ positive semi-definite on $\mathbf{R}_+^4$. The CPU time and RAM spent for the process are about 370 seconds and 680M on a P4 2.4G computer. So far we do not know any program else which could do this problem as well.

## 3 Successive Difference Substitution

Given a homogeneous polynomial $F$, what can we do if the Difference Substitution of $F$ is not trivially nonnegative, i.e. DS$(F)$ includes polynomials with some negative coefficients?

For example, to prove the 4-cyclic inequality,

$$\frac{a_1}{a_2 + a_3} + \frac{a_2}{a_3 + a_4} + \frac{a_3}{a_4 + a_1} + \frac{a_4}{a_1 + a_2} \geq 2, \tag{13}$$

we need show the following polynomial is positive semi-definite on $\mathbf{R}_+^4$,

$$\begin{aligned} F = {} & a_1^3a_3 + a_1^3a_4 + a_1^2a_2^2 - a_1^2a_2a_4 - 2\,a_1^2a_3^2 - a_1^2a_3a_4 + a_1^2a_4^2 + a_1a_2^3 - a_1a_2^2a_3 - a_1a_2^2a_4 \\ & - a_1a_2a_3^2 + a_1a_3^3 - a_1a_3a_4^2 + a_2^3a_4 + a_2^2a_3^2 - 2a_2^2a_4^2 + a_2a_3^3 - a_2a_3^2a_4 - a_2a_3a_4^2 + a_2a_4^3 \\ & + a_3^2a_4^2 + a_3a_4^3. \end{aligned} \tag{14}$$

Here DS$(F)$ consists 6 polynomials, one of which with some negative coefficients, namely,

$$\begin{aligned} F_1 = {} & t_1^3t_2 + t_1^3t_3 + 2\,t_1^3t_4 + t_1^2t_2^2 + 2\,t_1^2t_2t_3 + 4t_1^2t_2t_4 + 2\,t_1^2t_3^2 + 5\,t_1^2t_3t_4 + 4t_1^2t_4^2 - t_1t_2^2t_3 \\ & - t_1t_2t_3^2 - 2t_1t_2t_3t_4 + t_1t_3^3 + t_1t_3^2t_4 + t_2^2t_3^2 + 3t_2t_3^3 + 4t_2t_3^2t_4 + 2t_3^4 + 6t_3^3t_4 + 4t_3^2t_4^2, \end{aligned}$$

so it is not trivially nonnegative. We can prove, however, the nonnegativity of $F_1$ on $\mathbf{R}_+^4$ by use the splitting transformations again, for instance, the following one,

$$\begin{cases} t_1 = u_1 + u_2 + u_3 + u_4 \\ t_2 = u_2 + u_3 + u_4 \\ t_3 = u_3 + u_4 \\ t_4 = u_4, \end{cases} \tag{15}$$

corresponds to the order $t_1 \geq t_2 \geq t_3 \geq t_4$. We implemented 24 transformations by computer and obtained $\mathrm{DS}\,(F_1)$ which consists of 24 polynomials with nonnegative coefficients, so $F_1$ is positive semi-definite on $\mathbf{R}_+^4$, hence $F$ too.

Let us formally define the procedure by induction as follows.

- Given a polynomial $F$, compute the Difference Substitution, $\mathrm{DS}\,(F)$.
- Define $\mathrm{DS}_0(F) = \{F\}$, $\mathrm{DS}_1(F) = \mathrm{DS}\,(F)$.
- If the set $\mathrm{DS}_k(F)$ is trivially nonnegative, stop.
- Otherwise, denote the polynomials in $\mathrm{DS}_k(F)$ which are with some negative coefficients by $F_{k,1}, F_{k,2}, \cdots, F_{k,l_k}$, compute $\mathrm{DS}\,(F_{k,1})$, $\mathrm{DS}\,(F_{k,2})$, $\cdots$, $\mathrm{DS}\,(F_{k,l_k})$.
- Define $\mathrm{DS}_{k+1}(F) = \bigcup_{i=1}^{l_k} \mathrm{DS}\,(F_{k,i})$.
- If $\mathrm{DS}_{k+1}(F)$ is trivially nonnegative, stop. Otherwise, the procedure will continue.

The procedure may possibly never terminate even though the original polynomial $F$ is positive semi-definite. We have a short program named SDS (Successive Difference Substitution) written in Maple which just implements one step of the above procedure. For instance, to verify the nonnegativity of previous polynomial (14),

$$\begin{aligned} F = {} & a_1^3 a_3 + a_1^3 a_4 + a_1^2 a_2^2 - a_1^2 a_2 a_4 - 2\, a_1^2 a_3^2 - a_1^2 a_3 a_4 + a_1^2 a_4^2 + a_1 a_2^3 - a_1 a_2^2 a_3 - a_1 a_2^2 a_4 \\ & - a_1 a_2 a_3^2 + a_1 a_3^3 - a_1 a_3 a_4^2 + a_2^3 a_4 + a_2^2 a_3^2 - 2 a_2^2 a_4^2 + a_2 a_3^3 - a_2 a_3^2 a_4 - a_2 a_3 a_4^2 + a_2 a_4^3 \\ & + a_3^2 a_4^2 + a_3 a_4^3, \end{aligned}$$

the input `sds(sds(F))` creates an output: "*The form is positive semi-definite*". We used here program SDS twice.

`Problem 8.` Show that the following polynomial is positive semi-definite on $\mathbf{R}_+^3$,

$$H = x^4 y^2 - 2\, x^4 y z + x^4 z^2 + 3 x^3 y^2 z - 2\, x^3 y z^2 - 2\, x^2 y^4 - 2\, x^2 y^3 z + x^2 y^2 z^2 + 2\, x y^4 z + y^6.$$

The input `sds(sds(sds(sds(sds(H)))))` creates an output: "*The form is positive semi-definite*". Program SDS was used five times. We may use the recurrence command instead,

```
> for i to 5 do H:=sds(H) od:
```

`Problem 9.` Show that the following polynomial is positive semi-definite on $\mathbf{R}_+^3$,

$$\begin{aligned}F = {} & 8x^7 + (8z + 6y)x^6 + 2\,y(31y - 77z)x^5 - y(69y^2 - 2\,z^2 - 202\,yz)x^4 \\ & + 2\,y(9y^3 + 57yz^2 - 85y^2z + 9z^3)x^3 + 2\,y^2z(-13z^2 - 62\,yz + 27y^2)x^2 \\ & + 2\,y^3z^2(-11z + 27y)x + y^3z^3(z + 18y).\end{aligned} \tag{16}$$

The 18-step recurrence command

```
> for i to 18 do F:=sds(F) od:
```

creates an output: "*The form is positive semi-definite*". This problem comes from " `http://guestbook.nease.net/read.php?user=zgbdsyjxz&id=1118234222&curpage=35` ".

`Problem 10.` Show that the following polynomial is positive semi-definite on $\mathbf{R}^3$,

$$F = a(a + b)^5 + b(c + b)^5 + c(a + c)^5. \tag{17}$$

This problem is different from the previous ones for each of the variables, $a, b, c$, is allowed to be negative. According to the signs of $a, b, c$, we partition the problem into several instances to deal with respectively. Say, if $a \geq 0, b < 0, c < 0$, substitute $x, -y, -z$ for $a, b, c$ in $F$, we get a polynomial in nonnegative variables,

$$f_1 = x(x - y)^5 - y(-z - y)^5 - z(x - z)^5, \tag{18}$$

which can be proven to be positive semi-definite on $\mathbf{R}_+^3$ by the 4-step command,

```
> for i to 4 do f₁:=sds(f₁) od:
```

Other instances can be done analogously. The problem comes from `http://www.mathlinks.ro/Forum/topic-30448.html`

`Problem 11.` Show that the following polynomial is positive semi-definite on $\mathbf{R}_+^3$,

$$\begin{aligned}G = {} & 2572755344x^4 - 20000000x^3y - 6426888360x^3z + 30000000x^2y^2 + 5315682897x^2z^2 \\ & - 20000000xy^3 - 1621722090xz^3 + 170172209y^4 - 1301377672\,y^3z + 3553788598y^2z^2 \\ & - 3864133016yz^3 + 1611722090z^4.\end{aligned} \tag{19}$$

The polynomial is of degree 4 and has 12 terms only. It seems not to be a large polynomial anyway, however, the process could not terminate until the program SDS is used for 46 times!

```
> for i to 46 do G:=sds(G) od:
```

The 46-step recurrence command creates an output: "*The form is positive semi-definite*".

The method originated from a plain idea, split the variables into smaller nonnegative quantities, so the polynomials to be treated with are considered *homogeneous* ones, that is, the variables of the same polynomial should have equal dimensions. Although the program SDS is applicable to all polynomials, its performance to nonhomogeneous ones is often inefficient. So the right way is convert the given polynomial to a homogeneous one at first.

## 4 About Symmetric Forms

Presented in the paper is a heuristic method, not a complete algorithm. So far we cannot better define the set of such polynomials that the program SDS can successfully apply to. Approaches to this aspect are pending. We will, however, introduce in this section several relevant results about symmetric forms, i.e. homogeneous symmetric polynomials, without proofs.

A form $F$ is called *trivially nonnegative* if the set $\mathrm{DS}(F)$ is trivially nonnegative.

**Theorem 1.** A symmetric 3-degree form $f(x_1, x_2, \cdots, x_n)$ with $f(1, 1, \cdots, 1) = 0$ is positive semi-definite on $\mathbf{R}_+^n$ if and only if it is trivially nonnegative.

**Theorem 2.** A symmetric quaternary 4-degree form $G(x_1, x_2, x_3, x_4)$ with $G(1, 1, 1, 1) = 0$ is trivially nonnegative if and only if there exist 5 nonnegative numbers $a_1, a_2, a_3, a_4, a_5$ such that $G = a_1 g_1 + a_2 g_2 + a_3 g_3 + a_4 g_4 + a_5 g_5$ where

$$
\begin{aligned}
g_1 = {} & 3x_1^4 + 3x_2^4 + 3x_3^4 + 3x_4^4 - 4x_1^3(x_2 + x_3 + x_4) - 4x_2^3(x_1 + x_3 + x_4) - 4x_3^3(x_1 + x_2 + x_4) \\
& - 4x_4^3(x_1 + x_2 + x_3) + 2\,x_1^2(x_2^2 + x_3^2 + x_4^2) + 2\,x_2^2(x_3^2 + x_4^2) + 2\,x_3^2x_4^2 \\
& + 4x_1^2(x_2x_3 + x_3x_4 + x_2x_4) + 4x_2^2(x_1x_3 + x_3x_4 + x_4x_1) + 4x_3^2(x_1x_2 + x_2x_4 + x_4x_1) \\
& + 4x_4^2(x_1x_2 + x_2x_3 + x_1x_3) - 24x_1x_2x_3x_4, \\
g_2 = {} & x_1^2(x_2^2 + x_3^2 + x_4^2) + x_2^2(x_3^2 + x_4^2) + x_3^2x_4^2 - x_1^2(x_2x_3 + x_3x_4 + x_2x_4) \\
& - x_2^2(x_1x_3 + x_3x_4 + x_4x_1) - x_3^2(x_1x_2 + x_2x_4 + x_4x_1) - x_4^2(x_1x_2 + x_2x_3 + x_1x_3) \\
& + 6x_1x_2x_3x_4, \\
g_3 = {} & 3x_1^4 + 3x_2^4 + 3x_3^4 + 3x_4^4 - 2\,x_1^3(x_2 + x_3 + x_4) - 2\,x_2^3(x_1 + x_3 + x_4) - 2\,x_3^3(x_1 + x_2 + x_4) \\
& - 2\,x_4^3(x_1 + x_2 + x_3) - 2\,x_1^2(x_2^2 + x_3^2 + x_4^2) - 2\,x_2^2(x_3^2 + x_4^2) - 2\,x_3^2x_4^2 \\
& + 3x_1^2(x_2x_3 + x_3x_4 + x_2x_4) + 3x_2^2(x_1x_3 + x_3x_4 + x_4x_1) + 3x_3^2(x_1x_2 + x_2x_4 + x_4x_1) \\
& + 3x_4^2(x_1x_2 + x_2x_3 + x_1x_3) - 12x_1x_2x_3x_4, \\
g_4 = {} & x_1^2(x_2x_3 + x_3x_4 + x_2x_4) + x_2^2(x_1x_3 + x_3x_4 + x_4x_1) + x_3^2(x_1x_2 + x_2x_4 + x_4x_1) \\
& + x_4^2(x_1x_2 + x_2x_3 + x_1x_3) - 12x_1x_2x_3x_4, \\
g_5 = {} & x_1^3(x_2 + x_3 + x_4) + x_2^3(x_1 + x_3 + x_4) + x_3^3(x_1 + x_2 + x_4) + x_4^3(x_1 + x_2 + x_3) \\
& - 2\,x_1^2(x_2^2 + x_3^2 + x_4^2) - 2\,x_2^2(x_3^2 + x_4^2) - 2\,x_3^2x_4^2
\end{aligned}
$$

all are symmetric quaternary 4-degree forms.

**Theorem 3.** A symmetric ternary 5-degree form $F(x, y, z)$ with $F(1, 1, 1) = 0$ is trivially nonnegative if and only if there exist 8 nonnegative numbers $b_1, b_2, \cdots, b_8$ such that

$$F = b_1 f_1 + b_2 f_2 + \cdots + b_8 f_8 \quad \text{where} \tag{20}$$

$$
\begin{aligned}
f_1 = {} & x^5 + y^5 + z^5 - x^4(y + z) - y^4(x + z) - z^4(x + y) + xyz(x^2 + y^2 + z^2), \\
f_2 = {} & x^4(y + z) + y^4(x + z) + z^4(x + y) - x^3(y^2 + z^2) - y^3(x^2 + z^2) - z^3(x^2 + y^2) \\
& - 2\,xyz(x^2 + y^2 + z^2) + 2\,xyz(xy + yz + xz), \\
f_3 = {} & x^3(y^2 + z^2) + y^3(x^2 + z^2) + z^3(x^2 + y^2) - 2\,xyz(x^2 + y^2 + z^2), \\
f_4 = {} & x^3yz + xy^3z + xyz^3 - x^2y^2z - xy^2z^2 - x^2yz^2,
\end{aligned}
$$

$$f_5 = x^5 + y^5 + z^5 - 2\,x^4y - 2\,x^4z - 2\,y^4x - 2\,y^4z - 2\,z^4x - 2\,z^4y + x^3y^2 + x^3z^2 + y^3x^2 + y^3z^2 + z^3x^2 + z^3y^2 + 4x^3yz + 4xy^3z + 4xyz^3 - 3x^2y^2z - 3xy^2z^2 - 3x^2yz^2,$$

$$f_6 = x^4y + x^4z + y^4x + y^4z + z^4x + z^4y - 8x^3yz - 8xy^3z - 8xyz^3 + 6x^2y^2z + 6xy^2z^2 + 6x^2yz^2,$$

$$f_7 = 2\,x^5 + 2\,y^5 + 2\,z^5 - 5x^4y - 5x^4z - 5y^4x - 5y^4z - 5z^4x - 5z^4y + 3x^3y^2 + 3x^3z^2 + 3\,y^3x^2 + 3\,y^3z^2 + 3z^3x^2 + 3z^3y^2 + 14x^3yz + 14xy^3z + 14xyz^3 - 12\,x^2y^2z - 12\,xy^2z^2 - 12\,x^2yz^2,$$

$$f_8 = 3\,x^4(y+z) + 3\,y^4(x+z) + 3z^4(x+y) - 2\,x^3(y^2+z^2) - 2\,y^3(x^2+z^2) - 2\,z^3(x^2+y^2) - 14xyz(x^2+y^2+z^2) + 12\,xyz(xy+yz+xz)$$

all are symmetric ternary 5-degree forms.

## 5 Conclusion

Based on a plain idea originated with nameless people maybe, we develop a heuristic to prove polynomial inequalities, or equivalently, to decide the nonnegativity of polynomials. That is, split the variables into smaller nonnegative quantities by so-called difference substitution, then, collect the terms and see whether the coefficients all are nonnegative. The theoretical approach is at the beginning; however, the method was illustrated to be efficient sometimes by experiential results.

There are various packages for real algebra such as REDLOG [2], QEPCAD [1], BOTTEMA [4] and DISCOVERER [5] which may be applied to inequality-proving in automatic mode, but none of them managed to prove inequalities with more variables and higher degrees like those in problem 4 and problem 7.

Moreover, the method of difference substitution employs a little mathematics only, so it can be understood and accepted much more easily.

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