New Linear Models on RDB

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Abstract: In this paper we establish several new linear models on RDB. Let a ruler have n marks, and let its length be L. A set of models of marks distribution, each of which makes the ruler perfect, is denoted by *M*(L, *n*). Our main results are the following five linear models:

$$\{1, 1, 6, 7, 1, 10, 10, \mathbf{L}, 10, 3, 4, 2, 3\} \in M(10n-52, n)$$
$$\{1, 3, 1, 4, 4, 11, 11, \mathbf{L}, 11, 3, 7, 6, 1, 1\} \in M(11n-68, n)$$
$$\{1, 3, 1, 5, 1, 4, 12, 12, \mathbf{L}, 12, 6, 2, 5, 6, 2\} \in M(12n-84, n)$$
$$\{1, 1, 2, 1, 1, 5, 13, 13, \mathbf{L}, 13, 4, 8, 2, 7, 7, 1\} \in M(13n-103, n)$$
$$\{1, 1, 1, 1, 5, 2, 14, 14, \mathbf{L}, 14, 6, 7, 2, 4, 7, 5, 1\} \in M(14n-125, n)$$

By the new models, in applicable range of n (<30) we have improved the author's previous results, which were derived from the quadratic models.

1. Introduction

Let K_n be a labeled complete graph of order n. Then the mark of an edge in K_n is defined as the difference of the marks of the two vertices incident with it. It is well known that when n>4 there does not exist any graceful labeling in a complete graph K_n. In other words, we are not able to label the vertices in such a way that the marks of edges just take 1 through c_n^2 without repetition. If there exists a labeling of vertices such that the edges are marked with successive integers from 1 to $G(K_n)$ and $G(K_n)$ is as big as possible, we call the set of the marks of the vertices of K_n an **RDB** (**restricted difference basis**).

RDB is widely used in many areas, such as test technology and network. In the research of RDB, we adopt the ruler model: The vertex marked as d in K_n corresponds to a mark on the ruler and the distance between the mark and the started end of the ruler is d. Hence the distance between two marks on the ruler is the difference between the two corresponding vertex marks. An RDB ruler (which is also called "the sparse ruler") is **perfect** if it can measure any integer length between 1 to the length of the ruler [1]-[10].

In this paper we adopt the following definitions and notations which have appeared in [11]-[13]: Let a ruler have n marks (excluding the two marks of its ends), and let its length be L. A set of models of marks distribution, each of which makes the ruler perfect, is denoted by *M*(L, *n*). The distance between any two adjacent marks of the ruler is called a **segment**. If the length of a segment is x we call the segment an "x" segment denoted by "x". Successive *u* "x" segments is called an "x" **block** denoted by D(x, u). We have presented two common RDB models of marks distribution in [11]-[13]: linear models and quadratic models. We call a model **increment t** model if it has the form

$$\{b_1, \ldots, b_r, t, \ldots\ldots, t, b_{r+1}, \ldots, b_{t-1}\} \in M(tn-s,\ n), \tag{I}$$

where the left hand side is the sequence of segments generated by the marks dividing the ruler and each of the t-1 segments $b_1, \ldots, b_r,\ b_{r+1}, \ldots, b_{t-1}$ is less than t. Clearly, there are n+1 segments among which just n-t+2 segments are "t" segments. We say the increment model is a linear model since in it the length L of the ruler is a linear function in n.

In 1996 we obtained "increment 6" and "increment 7" linear models in [11]:

$$\{1, 3, 6, 6, \ldots, 6, 2, 3, 2\} \in M(6n-13,\ n) \quad (n>4). \tag{1}$$

$$\{1, 2, 3, 7, 7, \ldots, 7, 4, 4, 1\} \in M(7n-20,\ n) \quad (n>5) \tag{2}$$

In 2004 we obtained "increment 8", "increment 9" and "increment 10" linear models in [13]:

$$\{1, 4, 5, 8, 8, \ldots, 8, 3, 1, 2, 1\} \in (8n-31, n) \quad (n>6) \tag{3}$$

$$\{1, 4, 3, 4, 9, 9, \ldots, 9, 5, 1, 2, 2\} \in M(9n-41, n) \quad (n>7) \tag{4}$$

$$\{1, 1, 3, 4, 10, 10, \ldots, 10, 2, 3, 6, 2, 3\} \in M(10n-55, n) \quad (n>8) \tag{5}$$

We call a model an "**x link**" distribution model if it has the form

$$D(x, u)+ D(1, x-1)+m\{\text{"}t\text{"}+ D(1, x-1)\}+\text{"}r\text{"}+ D(1, x-1) \quad M(L, n). \tag{II}$$

In [12] we established "x link" distribution models and improved Miller's conclusion in [3]. We say an "x link" distribution model is a quadratic model since in it the length L of the ruler is a quadratic function in n.

For example, for n=t=6 the ruler corresponding the "increment t" model is as bellow:

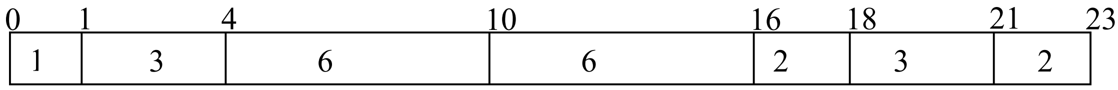

Figure 1.1 The ruler for n=t=6

In this paper, array coverage is applied to obtain several newer linear distribution models than "increment 6" model and "increment 10" model.

2. Exploration of Array Coverage

In order to explore the linear models of RDB, we assume that the ruler's length is L and there are n+2 marks on the ruler as in the last section. The i^{th} mark is denoted by a(i) (i=0,…,n+1), where a(0)=0 and a(n+1)=L are the two marks of the ends of the ruler, and a(i) (0<i<n+1) is the integer length between the i^{th} mark and the 0^{th} mark. We apply the expression (I) in Section 1 to explore "increment t" linear models. An array *b* is set up to denote the all partial sums of successive segments. Since the distance between any two marks is just the partial sum of all the successive segments between the two marks, we obtain C_{n+2}^2 =(n+2)(n+1)/2 partial sums of successive segments, which are all the lengths that can be measured by the ruler. If these sums can cover the integer numbers within L completely, the obtained distribution models can make the ruler perfect. Here follows the program of the array coverage design:

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 y=(n+1)*(n+2)/2;
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for (k=0,i=0;i<=n; i++)
   {for(j=i+1;j<=n+1;j++)  {k++; b[k]=a[j]-a[i];}}
/* get the y partial sums of lengths of succeeding segments.  */
for(u=0,d=1;d<=L; d++)        /* Check every integer d within L.*/
{for(i=1;i<=y; i++)
      {if (b[i]==d)  {u=u+1;i=y;}}}
/* If the partial sum of successive segments equals d, u is increased by 1. */
   if (u==L)     /* if u=L, complete coverage is achieved, print to output */
   for (i=1;i<=n+1; i++) print a[i];
………………………………
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For given number of marks n and given t, we run the program repeatedly under different lengths L of ruler to seek the distribution result for the biggest L. New linear models on RDB ruler can be obtained, provided that the application of such distribution to general m segments of length t is proved.

3. Main Result and Proof

We obtain the following new linear models according to the program of array coverage design:

Theorem Any one of the following linear distribution models of "increment 10", "increment 11", "increment 12", "increment 13" and "increment 14" can make the ruler perfect:

1) the distribution model of "increment 10"

$$\{1,1,6,7,1,10,10,\mathbf{L},10,3,4,2,3\} \in M(10n-52,n) \quad (6)$$

2) the distribution model of "increment 11":

$$\{1,3,1,4,4,11,11,\mathbf{L},11,3,7,6,1,1\} \in M(11n-68,n) \quad (7)$$

3) the distribution model of "increase 12":

$$\{1,3,1,5,1,4,12,12,\mathbf{L},12,6,2,5,6,2\} \in M(12n-84,n) \quad (8)$$

4) the distribution model of "increase 13":

$$\{1,1,2,1,1,5,13,13,\mathbf{L},13,4,8,2,7,7,1\} \in M(13n-103,n) \quad (9)$$

5) the distribution model of "increase 14":

$$\{1,1,1,1,5,2,14,14,\mathbf{L},14,6,7,2,4,7,5,1\} \in M(14n-125,n) \quad (10)$$

Proof. We only need to prove the distribution (10) makes the ruler perfect, so does the any other distribution.

In (10), let there be just m segments of length 14. We need only to prove the ruler can measure any integer between 14c+1 and 14c+13 (c=1,2,…,m), that is, all the partial sums of successive segments can cover all the integers from 14c+1 to 14c+13. For convenience, we denote c successive segments of length 14 by {14} (c=0,1,2,…,m).

The proof can be finished by observing the following table (In which c=1,2,…,m except special explanation):

Table 3.1 The length the ruler can measure

The sum of segments' lengths	The length the ruler can measure
{14}+6+7+2	The ruler can measure 14c+1
2+{14}	The ruler can measure 14c+2
4+7+5+1	The ruler can measure 14c+3,(c=1)
{14}+6+7+2+4+7+5	The ruler can measure 14c+3,($2 \le c \le m$)
2+4+5+7	The ruler can measure 14c+4,(c=1)
{14}+6+7+2+4+7+5+1	The ruler can measure 14c+4,($2 \le c \le m$)
{14}+6+7+2+4	The ruler can measure 14c+5
{14}+6	The ruler can measure 14c+6
5+2+{14}	The ruler can measure 14c+7
1+5+2+{14}	The ruler can measure 14c+8
1+1+5+2+{14}	The ruler can measure 14c+9
1+1+1+5+2+{14}	The ruler can measure 14c+10
1+1+1+1+5+2+{14}	The ruler can measure 14c+11
{14}+6+7+2+4+7	The ruler can measure 14c+12
{14}+6+7+2+4+7+1	The ruler can measure 14c+13

Therefore, we have proved that "increment 14" distribution model can make the ruler perfect. Besides, these linear models are beautifully arranged and ingeniously conceived to meet the continuity of RDB.

By the way, we point out that the "increment 10" distribution model given in formula (6) is superior to that in formula (5), because for the same number "n" the range of measurement of formula (6) is 3 units longer than that of formula (5).

Example 3.1 Let the number of marks be 16. What is the maximum length of the perfect ruler that the given distribution models can make? What is the RDB of the corresponding 18 vertices?

In the "increment 10" distribution models, taking n=16 we get

$\{1, 1, 6, 7, 1, 10, 10, 10, 10, 10, 10, 10, 10, 3, 4, 2, 3\} \in M(108,16)$.

In the "increment 11" distribution models, taking n=16 we get

$\{1, 3, 1, 4, 4, 11, 11, 11, 11, 11, 11, 11, 3, 7, 6, 1, 1\} \in M(108, 16)$

In the "increment 12" distribution models, taking n=16 we get

$\{1, 3, 1, 5, 1, 4, 12, 12, 12, 12, 12, 12, 6, 2, 5, 6, 2\} \in M(108,16)$

It is easy to know that the maximum length obtained by other models is less than 108, so when the number of marks is 16, we should apply "increment 10", "increment 11" and "increment 12" linear models to make the ruler perfect, with the maximum length 108. The RDB's of 18 vertices corresponding the above 3 models are:

$\{0, 1, 2, 8, 15, 16, 26, 36, 46, 56, 66, 76, 86, 96, 99, 103, 105, 108\}$,

$\{0, 1, 4, 5, 9, 13, 24, 35, 46, 57, 68, 79, 90, 93, 100, 106, 107, 108\}$

and $\{0, 1, 4, 5, 10, 11, 15, 27, 39, 51, 63, 75, 87, 93, 95, 100, 106, 108\}$.

4. Comparison between Linear Models and Quadratic Models

In [11],[12], we have established "x link" quadric models of RDB. For x=1, we get L=(m+2)(n+2-m)-3, and the maximum length obtained by "1 link" model is

$$len = \left(\frac{n+4}{2}\right)^2 - \left(\frac{n}{2} - m\right)^2 - 3. \tag{11}$$

For x=2, we get L=(2m+4)(n+1-2m)-8, and the maximum length obtained by "2 link" model is

$$len = \left(\frac{n+5}{2}\right)^2 - \left(\frac{n-4m-3}{2}\right)^2 - 8. \tag{12}$$

Comparing "increment 14" with "2 link", we get $n \le 29$ by solving. Hence, linear models (1)-(5) are superior to quadric models provided n is in the application interval, and n is not big enough. It can be illustrated in the following table:

Table 4.1 Optimized intervals of linear models and "2 link" models

Intervals of n	Distribution models of maximum length	Formulas
$5 \le n \le 7$	"increment 6" models	len=6n-13
$7 \le n \le 10$	"increment 7" models	len=7n-20
$11 \le n \le 16$	"increment 10" models	len=10n-52
$16 \le n \le 19$	"increment 12" models	len=12n-84
$19 \le n \le 22$	"increment 13" models	len=13n-103
$22 \le n \le 29$	"increment 14" models	len=14n-125
$30 \le n$	"2 link " distribution models	$L = \left(\frac{n+5}{2}\right)^2 - \left(\frac{n-4m-3}{2}\right)^2 - 8$

Example 4.1 For n=23, applying "increment 13" linear model we get

$$\{1, 1, 2, 1, 1, 5, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 4, 8, 2, 7, 7, 1\} \in M(196, 23),$$

applying "increment 14" linear model we get

$$\{1, 1, 1, 1, 5, 2, 14, 14, 14, 14, 14, 14, 14, 14, 14, 14, 14, 14, 6, 7, 2, 4, 7, 5, 1\} \in M(197, 23),$$

applying "1 link" quadric model we get

$$\{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 12\} \in M(179, 23),$$

and applying "2 link" quadric models, we get

$$\{2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 27, 1, 27, 1, 27, 1, 27, 1, 27, 1, 24, 1\} \in M(188, 23).$$

Obviously, for n=23 linear models "increase 13" and "increase 14" are superior to "1 link" and "2 link" quadric models. The RDB of the 25 vertices corresponding to "increment 14" model is:

$$\{0, 1, 2, 3, 4, 9, 11, 25, 39, 53, 67, 81, 95, 109, 123, 137, 151, 165, 171, 178, 180, 184, 191, 196, 197\}.$$

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