Edge-bandwidth of Tensor Product of Paths and Cycles

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Abstract

The bandwidth of a graph is the minimum of the maximum difference between labels of adjacent vertices in the graph. If we label the edges instead of the vertices of the graph, we can define the edge-bandwidth accordingly. People start working on the edge-bandwidth of graphs since 1999[7]. The edge-bandwidth of a graph is the minimum of the maximum difference between labels of adjacent edges in the graph. Since the edge-bandwidth of a graph G is equal to the bandwidth of the line graph of G, establishing the edge-bandwidth of a graph is equivalent to verifying the bandwidth of one or more graphs. The decision problem corresponding to find the bandwidth of an arbitrary graph is NP-complete[10]. It is NP-complete even for trees of maximum degree 3[6]. Although the edge-bandwidth problem is included in the bandwidth problem, the computing complexity of the edge-bandwidth is unknown up to now. The application about the edge-bandwidth is in the area of network circuit on-line routing and admission control problem. The edge-bandwidth problem has been solved for only a few classes of graphs such as complete graph, complete bipartite graph with equal partites, caterpillar and theta graph[5]. This paper establishes the edge-bandwidth of the tensor product of a path with a path and a path with a cycle. Optimal edge-numberings to achieve each of these edge-bandwidths are provided.

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Key words: bandwidth, edge-bandwidth, tensor product, path, cycle.

1. Introduction and Terminology

For a graph G, $V(G)$ denotes the set of vertices of G and $E(G)$ denotes the set of edges of G. Bandwidth on graphs, and the analogous problem of bandwidth on matrices, has been studied since early 1950s (see [2]). The bandwidth of a graph is the minimum of the maximum difference between labels of adjacent vertices in the graph. If we label the edges instead of the vertices of the graph, we can define the edge-bandwidth accordingly.

Let $G = (V, E)$ be a graph on m edges, an injection function g: $E \rightarrow \{1,2,...,m\}$ is called a proper edge-numbering of G. The edge-bandwidth $B'_g(G)$ of a proper edge-numbering g of G is the number $B'_g(G) = \min\{|g(e_1) - g(e_2)| : e_1 \text{ and } e_2 \text{ are incident with a common vertex}\}$, and the edge-bandwidth $B'_g(G)$ of G is the number

$$B'(G) = \min\{B'_g(G) : g \text{ is a proper edge-numbering of } G\}.$$

Follow the definition [1], we have the line graph $L(G)$ of a graph G be the graph whose vertices can be put in one-to-one correspondence with the edges of G in such a way that two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent. Then $B'(G)$ is equal to the bandwidth of the line graph $L(G)$ of G, that is, $B'_g(G) = B(L(G))$.

A graph H is called a line graph if there exists a graph G such that $H = L(G)$. Since there may be more than one graph, say G_i, such that $H = L(G_i)$ for $i \geq 1$, hence establishing the edge-bandwidth of a graph is equivalent to verify the bandwidth of one or more graphs.

The decision problem corresponding to find the bandwidth of an arbitrary graph was shown to be NP-complete in [10]. In [6] it was shown that the problem is NP-complete even for trees of maximum degree 3. Although the bandwidth and the edge-bandwidth have certain relation, the computing complexity of the edge-bandwidth is unknown up to now.

The application about the edge-bandwidth is in the area of network circuit on-line routing and admission control problem. In [7] they proved that $B(G) \leq B'(G)$ for any graph G, and gave bounds on edge-bandwidth by adding, subdividing, or contracting edges in a graph. They also gave the edge-bandwidth of $K_n, K_{n,n}$, caterpillars, and some theta graphs. In [5] they discussed the edge-bandwidth of theta graphs. Here we turn our attention to the edge bandwidth of tensor product of paths and cycles.

2. Paths with Paths

Proposition 1 (from [3,4]) *If G is a connected graph and $D(G)$ is the diameter of G, then* $B(G) \geq \left\lceil \frac{|V(G)|-1}{D(G)} \right\rceil$.

Proposition 2 (from [11]) *Let $G = G_1(T_p)G_2$ for connected graphs G_1 and G_2. Then G is connected if and only if G_1 or G_2 has an odd cycle.*

Proposition 3 (from [9]) *Let* $G = G_1(T_p)G_2$ *for connected graphs* G_1 *and* G_2. *Then G consists of exactly two components if and only if* G_1 *and* G_2 *are both bipartite.*

Theorem 1 *Let* $G = P_m(T_p)P_n$ *with* $2 \le m \le n$. *Then* $B'(G) = \begin{cases} 1 & \text{if } m = 2, \\ m & \text{if } m \ge 3. \end{cases}$

Proof: When $m = 2$, *G* becomes two disjoint paths, so $B'(G) = 1$.

Now we consider the case where $m \ge 3$. Then, by Proposition 3, *G* must consist of exactly two components. Let G_1 be the component containing vertex v_{11}, G_2 be the component containing vertex v_{12}; and let $L(G_1)$ be the line graph of G_1, $L(G_2)$ be the line graph of G_2. Then $B'(G) = \max\{B(L(G_1)), B(L(G_2))\}$. Since $|V(L(G_1))| = |V(L(G_2))| = (m-1)(n-1)$ and $D(L(G_1)) = D(L(G_2)) = n-2$, by Proposition 1, $B(L(G_1)) = B(L(G_2)) \ge \left\lceil \frac{(m-1)(n-1)-1}{n-2} \right\rceil = m$ for $m \ge 3$. Consider *f*, a simple column by column sequential numbering of $L(G_1)$ and $L(G_2)$. Then $B_f(L(G_1)) = B_f(L(G_2)) = m$. Therefore we have $B'(G) = m$. Figure 1 and Figure 2 show the proper numbering of $L(G_1)$ and $L(G_2)$ for both *m*, *n* are odd. Figure 3 shows the proper numbering of $L(G_1)\left(L(G_1) \equiv L(G_2)\right)$ for other cases. That completes the proof. ❑

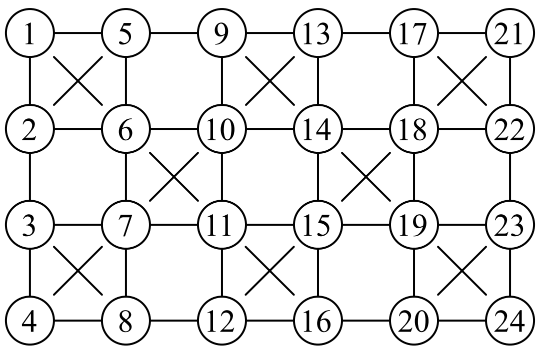

Figure 1: Proper numbering of $L(G_1)$ of $P_5(T_p)P_7$

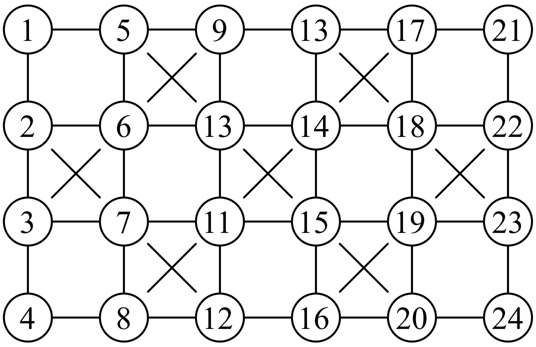

Figure 2 : Proper numbering of $L(G_2)$ of $P_5(T_p)P_7$

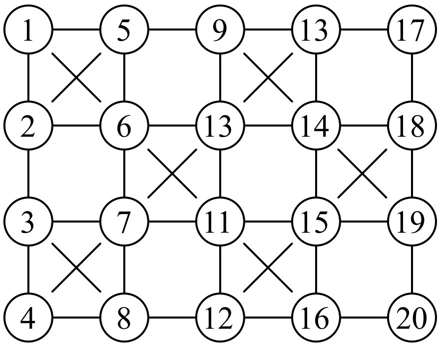

Figure 3 : Proper numbering of $L(G_1)$ of $P_5(T_p)P_6$

3. Paths with Cycles

For $S \subseteq V(G)$, let $\overline{S}$ denote $V(G)-S$ and ∂S denote the set of vertices in S adjacent to those in $\overline{S}$. For a proper numbering *f*, let $S_{f,t} = \{v \in V(G) : f(v) \le t\}$.

Proposition 4 (from [8]) *Let f be an optimal proper numbering of a connected graph G with n vertices. Then for* $t \in \{1,2,...,n\}$, $|\partial S_{f,t}| \le B(G)$.

Proposition 5 (from [8]) *Let f be an optimal proper numbering of a connected graph G with n vertices. Then for* $t \in \{1,2,...,n\}$, $|\partial \overline{S_{f,t}}| \le B(G)$.

Theorem 2 *Let* $G = P_m(T_p)C_n$ *with* $m \ge 2,\ n \ge 4$ *where n is even. Then*

$$B'(G) = \begin{cases} 2 & \text{if } m = 2, \\ 4 & \text{if } m = 3 \text{ and } n = 4, \\ 5 & \text{if } m = 3 \text{ and } n \ge 6, \\ \min\{2m-1, n+2\} & \text{if } m \ge 4. \end{cases}$$

Proof: Let R_i be row i of G, $R_i = \{(i, j) : 1 \le j \le n \text{ and } (i, j) \in G\}$, C_j be column j of G, $C_j = \{(i, j) : 1 \le i \le m \text{ and } (i, j) \in G\}$. By Proposition 3 we know G has exactly two components. Let G_1 be the component of G containing $(1,1)$, G_2 be the component of G containing $(1,2)$. We know that $G_1 \equiv G_2$. Let $L(G_1)$ be the line graph of G_1, then $B'(G) = B(L(G_1))$.

For $m = 2$, G_1 is a cycle, so $B'(G) = B(C_n) = 2$. For $m = 3$ and $n = 4$, define R_i' as row i of $L(G_1)$ to be $R_i' = \{(i, j) : 1 \le j \le n \text{ and } (i, j) \in V(L(G_1))\}$, and define C_j' as column j of $L(G_1)$ to be $C_j' = \{(i, j) : 1 \le i \le m-1 \text{ and } (i, j) \in V(L(G_1))\}$ (Figure 4 shows an example of $L(G_1)$ of $P_4(T_p)C_6$). Consider a proper numbering f_1 of $L(G_1)$ as in Figure 5. Then $B_{f_1}(L(G_1)) = 4$. Since every vertex in $L(G_1)$ has degree 4, we have $\left|\partial \overline{S_{f,1}}\right| = 4$ for all proper numbering f of $L(G_1)$. By Proposition 3.5, we have $B(L(G_1)) \ge 4$. Thus $B'(G) = B(L(G_1)) = 4$.

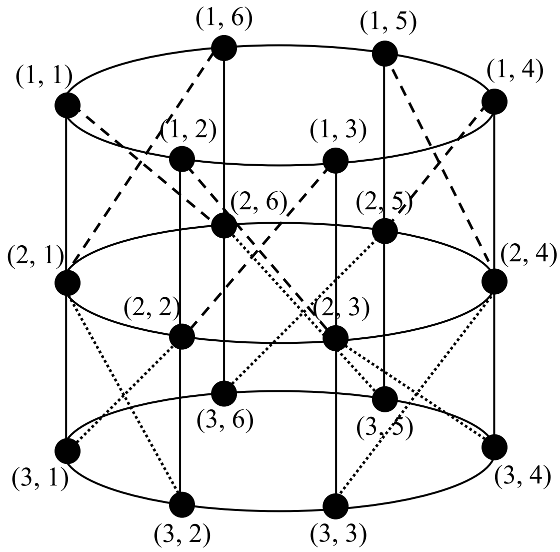

Figure 4 : $L(G_1)$ of $P_4(T_p)C_6$

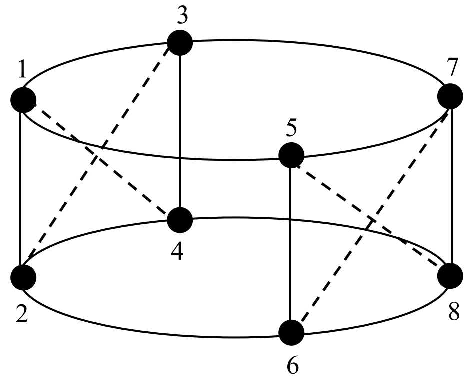

Figure 5 : Proper numbering of $L(G_1)$ from $P_3(T_p)C_4$

For $m=3$ and $n \geq 6$, consider proper numbering f_2 that numbers $L(G_1)$ as follows:

$$f_2(i, j) = \begin{cases} i + 4(j-1) & \text{if } 1 \leq i \leq 2 \text{ and } 1 \leq j \leq \dfrac{n}{2}, \\ 2(n-1) + i - 4(j - \dfrac{n}{2} - 1) & \text{if } 1 \leq i \leq 2 \text{ and } \dfrac{n}{2} + 1 \leq j \leq n. \end{cases}$$

Then $B_{f_2}(L(G_1)) = 5$ (Figure 6 is an example for the proper numbering of $L(G_1)$ of $P_3(T_p)C_6$).

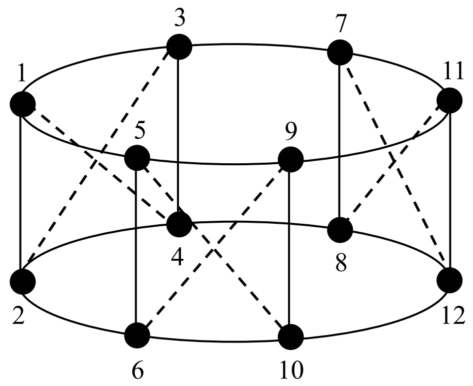

Figure 6 Proper numbering of $L(G_1)$ from $P_3(T_p)C_6$

Since we have either $\left|\partial S_{f,7}\right| \geq 5$ or $\left|\partial \overline{S_{f,7}}\right| \geq 5$ for all proper numbering f of $L(G_1)$, by Proposition 3.4 and Proposition 3.5 we have $B(L(G_1)) \geq 5$. Thus $B'(G) = B(L(G_1)) = 5$.

For $m \geq 4$, consider the following two cases:

Case 1: Suppose $n \geq 2m-2$. Define proper numbering g_1 of $L(G_1)$ that numbers $L(G_1)$ as follows:

$$g_1(i, j) = \begin{cases} i + 2(m-1)(j-1) & \text{if } 1 \leq i \leq m-1 \text{ and } 1 \leq j \leq \dfrac{n}{2}, \\ (m\text{-}1)(n-1) + i - 2(m-1)(j - \dfrac{n}{2} - 1) & \text{if } 1 \leq i \leq m-1 \text{ and } \dfrac{n}{2} + 1 \leq j \leq n. \end{cases}$$

Then $B_{g_1}(L(G_1)) = 2m-1$ (Figure 7 is an example for the proper numbering of $L(G_1)$ of $P_4(T_p)C_8$).

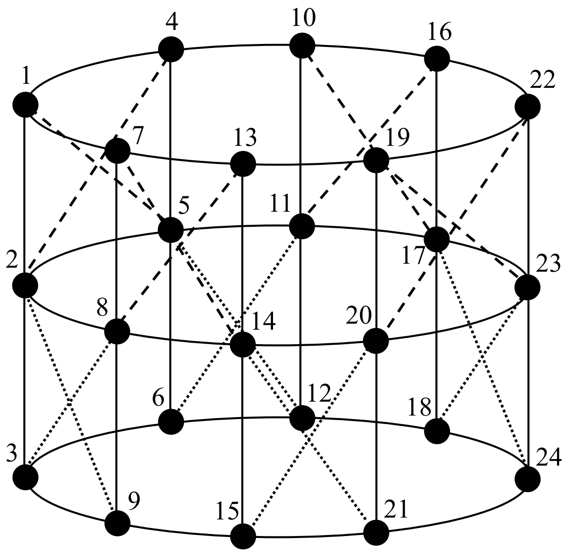

Figure 7 Proper numbering of $L(G_1)$ from $P_4(T_p)C_8$

Since we have either $|\partial S_{f,3m\text{-}2}| \geq 2m-1$ or $|\partial \overline{S_{f,3m\text{-}2}}| \geq 2m-1$ for all proper numbering *f* of $L(G_1)$, by Proposition 4 and Proposition 5 we have $B(L(G_1)) \geq 2m-1$. Therefore, in case 1, $B'(G) = B(L(G_1)) = 2m-1$.

Case 2: Suppose $n \leq 2m-4$. Since we have either $|\partial S_{f,n+2}| \geq n+2$ or $|\partial \overline{S_{f,n+2}}| \geq n+2$ for all proper numbering f of $L(G_1)$, by Proposition 4 and Proposition 5, we have $B_1'(G) \geq n+2$. Define proper numbering g_2 of $L(G_1)$ as follows:

$$g_2(i,\, j) = \begin{cases} j + (i-1)(n+1) & \text{if } 1 \leq i \leq m \text{ and } 1 \leq j \leq n-1, \\ n & \text{if } i = 1 \text{ and } j = n, \\ (i-1)(n+1) & \text{if } 2 \leq i \leq n \text{ and } j = n. \end{cases}$$

Then $B_{g_2}(L(G_1)) = n+2$ (Figure 8 is an example for the proper numbering of $L(G_1)$ of $P_6(T_p)C_6$). Therefore, in case 2, $B'(G) = B(L(G_1)) = n+2$.

Combining case 1 and case 2, we have $B'(G) = 2m-1$ for $n \geq 2m-2$ and $B'(G) = n+2$ for $n \leq 2m-4$. Therefore, we have $B'(G) = \min\{2m-1, n+2\}$. ❑

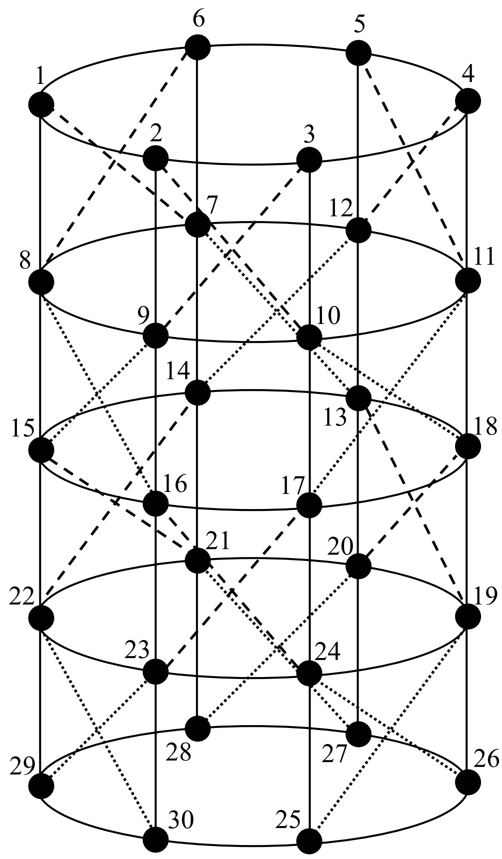

Figure 8 Proper numbering of $L(G_1)$ from $P_6(T_p)C_6$

Theorem 3 *Let* $G = P_m(T_p)C_n$ *with* $m \geq 2,\ n \geq 3$ *and n is odd. Then*

$$B'(G) = \begin{cases} 2 & \text{if } m = 2, \\ \min\{2m-1,\ 2n+2\} & \text{if } m \geq 3. \end{cases}$$

Proof: Since G has odd cycle, by Proposition 2, we know that G contains only one connected component. When $m = 2$, G becomes a cycle with $2n$ vertices, so $B'(G) = 2$.

Assume that $m \geq 3$. Let G_1 be the component of $P_m(T_p)C_{2n}$ that contains vertex $(1,1)$. We know that $G = P_m(T_p)C_n$ is isomorphic to G_1. Therefore, by Theorem 2, we have $B'(G_1) = \min\{2m-1,\ (2n)+2\} \Rightarrow B'(G) = \{2m-1,\ 2n+2\}$. ❑

4. Conclusions

The edge-bandwidth problem is a restricted version of the bandwidth problem. The edge-bandwidth of a graph *G* is equal to the bandwidth of the line graph of *G*. Although the bandwidth and the edge-bandwidth are related, the computing complexity of the edge-bandwidth is unknown up to now. Although the edge-bandwidth problem of graphs has been proposed for over ten years, but only very few classes of graphs that have been studied. This paper establishes the edge-bandwidth of the tensor product of a path with a path, and the edge-bandwidth of the tensor product of a path with a cycle for which bandwidth problem has been solved [8]. We get the lower bound of the edge-bandwidth for the tensor product of a path with a path, and the tensor product of a path with a cycle by using some known theorems for bandwidth of graphs then giving algorithms to find the exact edge-bandwidth for the corresponding graphs.

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