Some Examples of
Dynamic Proofs without Words in PowerPoint

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Abstract:

It is a crucial task to stimulate and to encourage students' interests on mathematical thinking as well as mathematical proving of certain mathematical facts by providing a visible environment, so that the students can catch related mathematical ideas with the help of their own visual intuition. The column of "Proofs without Words（PWWs 圖說證明）" has served this goal well over the past thirty years. Indeed, PWWs are pictures or diagrams that help the students to see *why* a particular statement may be true, and also to see *how* one might begin to go about proving it true. Based on this understanding, the notion of *Dynamic Proofs without Words* is proposed in this paper, followed by some dynamically elegant visual demonstrations of certain mathematical ideas including AG inequality, sums of constant powers of consecutive integers, trigonometric identities including double-angle formulas. Using *GSP* and MathType as tools, the dynamic environments for PWWs are managed within PowerPoint, some of those static configurations found in the column of "Proofs without Words " have been transformed into dynamic presentations. All dynamic environments in a CD - ROM are available on request.

Keyword: dynamic proof without words, sums of constant powers of consecutive integers, AG inequality, trigonometric identities

1. Introduction

The training of rigid mathematical thinking as well as mathematical proving of certain theorems and formulas is an essential part in mathematical education. However, it is usually also the hard point for high school students. P. Halmos once pointed out that "he saw mathematics not as logic but as pictures…" and " to be a scholar of mathematics, you must be born with … the ability to visualize". Polya's "draw a figure …" is a pedagogic advice, and Einstein and Poincare's view that we should use our visual intuition are well known. It is an important task to stimulate and to encourage students' interests by providing a visible environment, so that the students catch the mathematical ideas with the help of their own visual intuition. What could be better for this purpose than a

pleasing illustration that made an important mathematical point? The column of "Proofs without Words（PWWs）" in " *Mathematics Magazine*", published by MAA, have served this purpose quite well for the past thirty years. PWWs are not really proofs; indeed, they are pictures or diagrams that help the students to see *why* a particular statement may be true, and also to see *how* one might begin to go about proving it true; an equation or two may appear in order to guide the students in this process. The emphasis is clearly on providing visual clues to the students to stimulate mathematical thought. Parts of those works have been compiled by R. B. Nelson in [1, 2]; moreover, a database [3] of this column is now available on web.

Based on this understanding, the notion of d*ynamic Proofs without Words* is proposed in this paper followed by some dynamically elegant visual demonstrations of certain mathematical ideas. The dynamic environments for PWWs are managed within PowerPoint with the assistance of GSP and MathType. All necessary figures are drawn in GSP; all mathematical equations are prepared in MathType. Selected figures together with related mathematical equations are then copied and pasted within PowerPoint, followed by its function of animation. It must be pointed out that we must put our intentions on the layout together with the presenting order in custom animation, it certainly is the key for the success of dynamic presentations. Some of those static configurations found in [2] have been transformed into dynamic presentations in PowerPoint which provides the students another channel to catch mathematical ideas in a straightforward and intuitive way.

We will first show how to set up the motion pictures, and to make masks through example of arithmetic and geometric (AG) inequality (Gallant [1 pp. 49]); users can then produce their works to fit their own purposes. The following formulas

$$\sum_{k=1}^{n}(1+2+3+...+k) = \frac{n(n+1)(n+2)}{6} ,$$

$$1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6} , \text{ and } 1^3 + 2^3 + ... + n^3 = (1+2+...+n)^2$$

will be treated in details in Section 3 for illustration purpose. It's interesting to point out that the left hand side of the third one is the sum of volumes of a sequence of cubes with consecutive integers as their lengths, and the right hand side is the area of a square with the sum of consecutive integers as its length; the key idea is then to transform cubes into squares with proper animations. Dynamic presentations of certain trigonometric equalities step by step will be provided in Section 4.

2. Arithmetic and Geometric Inequality

We will show how to apply the animation function of PowerPoint to present course materials used in class teaching. Some procedures will also be introduced to lower the technology barrier for most teachers by taking advantage of the popularity of PowerPoint; we hope that teachers can easily access them with a success in class to help their students to catch up those mathematical ideas with less difficulty.

To demonstrate how the function of animation of PowerPoint can be used to transform those paper-based static PWWs into dynamic PWWs, we will show the drawing of graphs, the setting of animation, and the making of masks. All these works of animations can be done in any "editor mode"; those files can also be modified to fit each individuals own demanding.

2-1 The masks

The goal of the masks is to cover those unnecessary parts on slides, which can be achieved by putting the same color as that of the background. To prepare the masks:

1. the color of the background is suggest to be replaced by another one during the making for easy recognition and manipulation; replace it back after finish.

2. to transform the background into the color of gray, select the"*Format*/*Background*" as you wish; draw a rectangle large enough to cover the object to be covered; set up its color identical with the background without lines;

3. to draw a circle with one color, but without lines; choose these two objects, copy, delete; "*Paste special*"/ "*Picture GIF*"/ "*Sure*"; open the picture toolbar, click "*Set Transparent Color*"; indicate the circle by the pointer, this circle becomes transparent; turn back to the original color in the background.
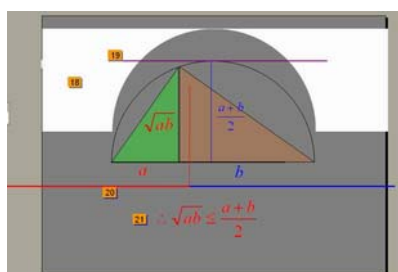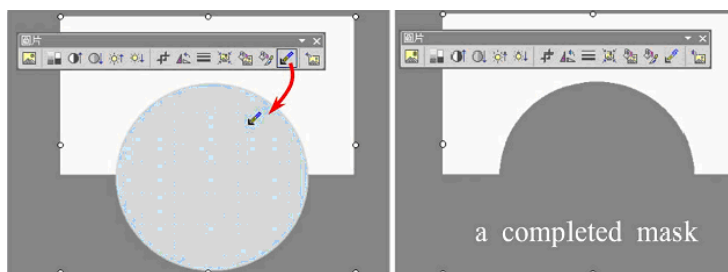

Figure 1

Figure 2 : a completed mask

2-2 The slides

The slides for introducing the arithmetic and geometric inequality $\sqrt{ab} \leq (a+b)/2$ are included for illustration purpose:

slide 1: draw $a, b, (a+b)/2, \sqrt{ab}$ and the half circle in order; show that the height from the meeting point to the semicircle is $\sqrt{ab}$ in terms of animation;

slide 2: to see the relationship between $\sqrt{ab}$ and $(a+b)/2$ when the lengths a, b are changed in animation;

slide 3: show that $\sqrt{ab} \leq (a+b)/2$; draw a line tangent to the top vertex of the semicircle and parallel to its diameter, then the distance between these two lines is $(a+b)/2$; to attract students on these two lines by fading these two triangles; move the red line right until it coincides with the blue line, then $a = b$ and the heights of these two lines segments are of equal length.

Keeping moving the red light to the most right, and then turn to move left; to convince the students that two line segments are of equal length if and only if $a = b$; and $\sqrt{ab} < (a+b)/2$ otherwise**;** the red line back to its original position and the two triangles fading gradually.
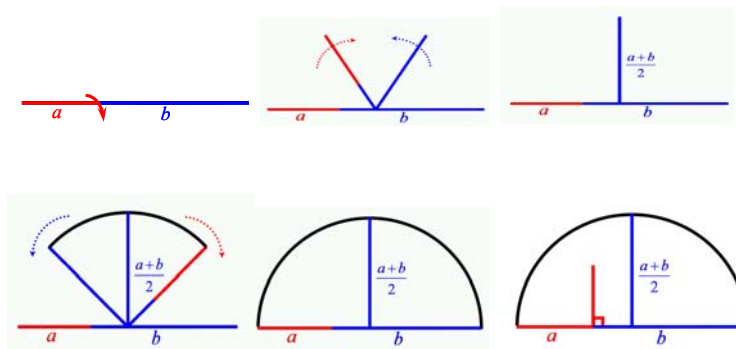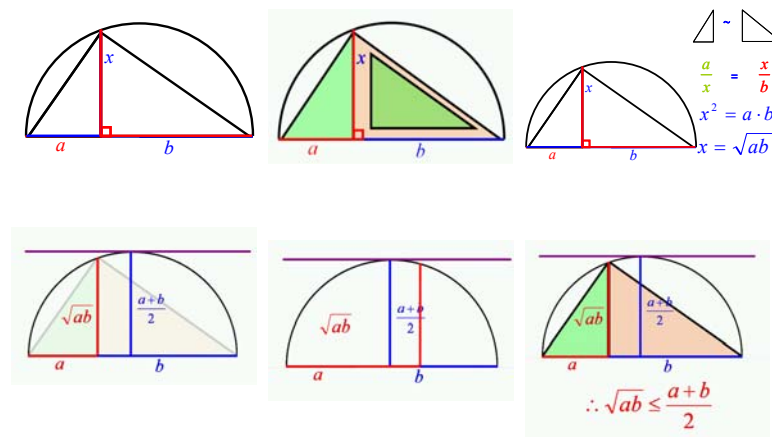
Figure 3: slide 1

Figure 4: slides 2 & 3

To show mathematical formulas in animation in PowerPoint 2002;

- · duplicate slide 1 and paste it on slide 2; delete those unnecessary parts; add new items and then number them continuously;
- · duplicate slide 2 and paste it on slide 3; delete those unnecessary parts; change the background into the color of gray for masks; recover it after finish;
- · indicate the relationship of $\sqrt{ab}$ and $(a+b)/2$ by "*From Left*", "*Erase*";
- · extend the lengths of red, blue lines to equal length; those groups over the half semicircle will be covered.

The height of the red line coincides with the blue radius in the order "*Motion Path*" / "*From Right*"; move next to the right of the semicircle by "*To Right*", and then "*AutoRecovery*", move the red height to the left side of the semicircle, and then back to its original position in the order of "*Motion Path* "/ "*Custom Path*; apply "*Erase*", "*From Left*" to finish.
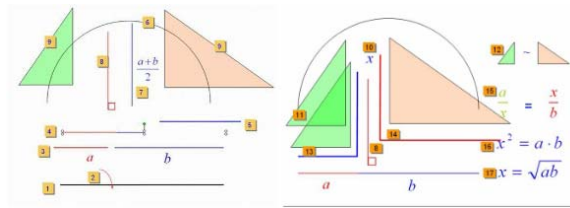
Figure 5

3. Sums of constant powers of consecutive integers

A few formulas for some of constant power of consecutive integers can be found in [3]; which deserve further study by taking advantage of nowadays advanced technology to transform those paper-based static PWWs into vivid dynamic presentation. We will briefly explain how to draw those squares, cubes, and animations; certain dynamic proofs are given for some of those formulas;

The following formulas

$$\sum_{k=1}^{n}(1+2+3+...+k) = \frac{n(n+1)(n+2)}{6},$$

$$1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}, \text{ and } 1^3 + 2^3 + ... + n^3 = (1+2+...+n)^2$$

will be considered in this section. The formula $1^3 + 2^3 + ... + n^3 = (1+2+...+n)^2$ is usually proved by the principle of mathematical induction in high school curriculum. It is interesting to point out that the left hand side of the third one is the sum of volumes of a sequence of cubes with consecutive integers as their length, and the right hand side is the area of a square with the sum of consecutive integers as its length. Because $\left(\sum_{k=1}^{n} k\right)^2 - \left(\sum_{k=1}^{n-1} k\right)^2 = n^3$, it follows that $\left(\sum_{k=1}^{n} k\right)^2 = \left(\sum_{k=1}^{n-1} k\right)^2 + n^3 = ... = \sum_{k=1}^{n} k^3$ recursively. The key idea is then how to transform cubes into squares.

The manipulations in PowerPoint are quite straightforward; for example, to perform next simply by pressing "*ENTER*" or "*PgDn*", to be back simply by pressing "*PgUp*", or to repeat simply by pressing "*ENTER*". We will show the processes of animations in this section to see the accumulations of cubes, 4-th power and 5-th power respectively. If $n = 1$, then $1^3 = 1^2$; if $n = 2$, $1^3 + 2^3 = (1 = 2)^3$; accumulate to one square; now 2^3 are added, we can see the process by pressing *ENTER*. The case $n = 3$ is similar to those cases as mentioned before, and hence all details are omitted.

Some newly added functions of PowerPoint XP can be used to produce squares, cubes, animation and to keep consistency of colors conveniently. Hold down <*Shift*> as you draw a selected object; a larger square is obtained from a small one by "*Copy/Paste*" a few times, the small square remains unchanged. For cubes, select a square, select "*Draw*", "*Change Autoshape*" , "*Basic Shapes*", and then select "*Cube*", to transform a square to a cube; the size of each face of the cube is identical

with the original one if start from a square with 133% of its original size. The objects can be moved to any position as wish through the four functions "*Entrance*", "*Exit* ", "*Motion Path* ", and "*Emphasis* " of animation provided in PowerPoint. For animation, select an object for moving; hold down the right key of the mouse to select custom animation and add effect based on individual demanding, and then those 4 effects will show up as shown. The most frequently used functions include "*Select Motion Path*", "*Draw Custom Path* (D)", line or curve as shown in figures. The positioning point of moving object can be located more precisely by "*Edit Point* " when "*Select Motion Path*" and "*Draw Custom Path*(D)".
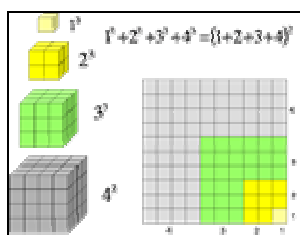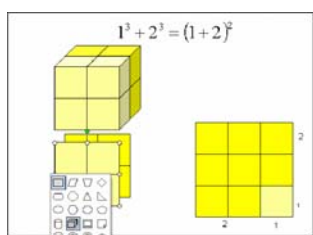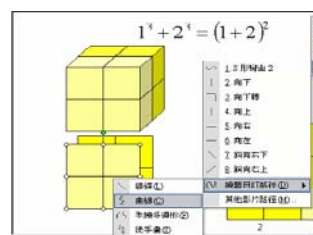

Fiure 6

Figure 7

Figure 8

4. Trigonometric identities

The subject of trigonometric functions is one of the major subjects in the curriculum in high school mathematics. Many trigonometric identities are also hard for high school students to realize and to memorize. How to put these pieces of figures made by GSP in PowerPoint so that those equalities can be presented step by step dynamically is what we concern in this section. Because the figures of GSP can be decomposed into some basic units in MS Word, GSP is used to draw all figures used for the proposed proofs. Mathematical equations are prepared in MathType. Selected figures together with related mathematical equations are then copied and pasted within PowerPoint, followed by the functions of animation. Some intentions must be put on the layout together with the presenting order in custom animation; indeed, it is the key for the success of dynamic presentations.

We must make it clear that the proofs of trigonometric identities within PWWs are under the restrictions that all angles lying between 0 and 90 due to technical difficulty, rather then the necessarily mathematical constraints. Some of those static configurations found in [2] have been transformed into their corresponding dynamic presentations in PowerPoint, which provides another channel for the students to catch those mathematical ideas in a straightforward and intuitive way.

After starting PowerPoint, choose a blank slide from "*Slide Layout*" / "*Content Layout*", paste the figure on it, modify to its proper size, then click the function "*Ungroup*"; and then the message will be shown, hold down "Y", and then hold down "*Ungroup*"once more, all figures made by GSP can be decomposed into the basic units as points, lines, planes or parts of circles for later use. Key in all mathematical statements and symbols in MathType, adjust them to their proper sizes, move them to their proper positions; same notation (expressions) can be prepared by duplication so that they are consistent; arrange the objects in its proper orders for better performance. We illustrate all these in the following example

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$$

step by step as follows, see Figure 9:

1. select a figure, copy and paste in a blank slide in PowerPoint, adjust to its proper size; keep the mouse at one of its corner when adjust its size, and hold down "*Shift*" to keep the ratio unchanged; after figure selected, click "Ungroup", executive twice; the graph can be decomposed into manageable basic units;
2. call MathType to type in some mathematical symbols like $\sin\theta$, $\tan\theta/2$; adjust to proper sizes, copy and paste in a few times, so that consistent mathematics symbols are available; once put them on proper positions, segments and arcs can be colored as wish or change their weights;
3. apply "*custom animation*", put all materials in proper order with special emphasis on the order of presentation of mathematical symbols.

Three other examples are included in Figures 10-12 for illustration purpose, all dynamic environments in a CD - ROM with more than 30 other examples are available on request.
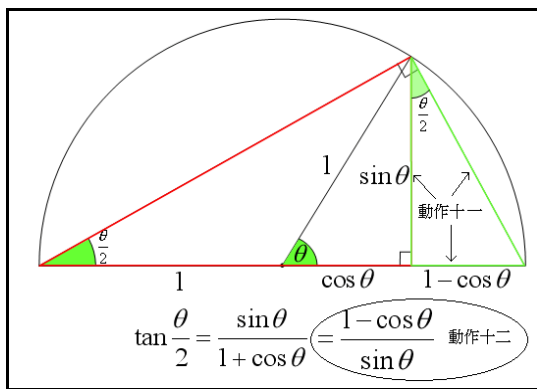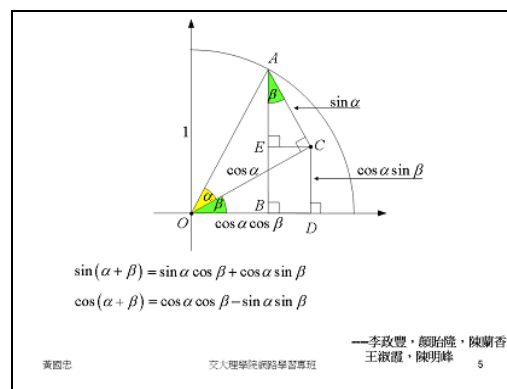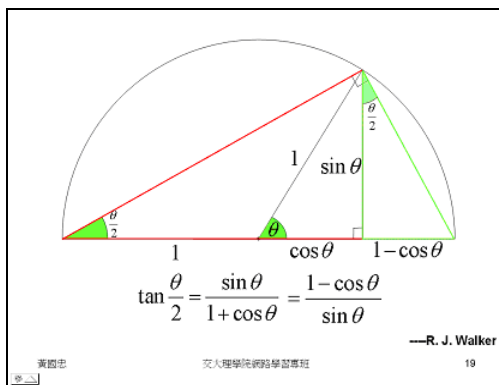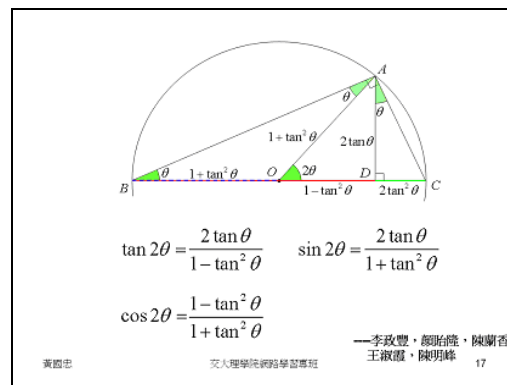

Figure 9:

Figure 10

Figure 11

Figure 12

References

[1] Roger B. Nelsen, *Proofs without Words － Exercises in Visual Thinking*, Classroom Resource Materials Number 1, The Mathematical Association of America (MAA) 1993.

[2] Roger B. Nelsen, *Proofs without Words II － More Exercises in Visual Thinking*, Classroom Resource Materials, the Mathematical Association of America (MAA) 2000.

[3] Mathematics Magazine, published by The Mathematics Association of America (MAA), http://www.maa.org or http://www.math.hmc.edu/MathMag.