The Optimal Design of Stable Inverse Transfer Function

Der-Feng Huang
Department of Electrical Engineering, Assistant professor
Lung-Hwa University of Science and Technology
300 Wan-shou Rd.,Sec.1,Kueishan Taoyuan, Taiwan
dfhuang@mail.lhu.edu.tw

Abstract

The design of the inverse system is a common issue on the design of a digital signal processing and communication systems. If the system transfer function is not minimum phase, the inverse system is unstable. Based on the least square error criterion, employing the calculation of variation technique and spectral factorization, a stable inverse transfer function is designed. Both FIR and IIR filter design examples are given to illustrate the design procedure.

Keyword: spectral factorization, stable inverse system

Introduction

There are many applications of the inverse system in the fields of digital signal processing, communication systems. The scheme of a signal transmission and receiving structure is shown in Fig. 1. If $h(z)$ is not minimum phase, the inverse system is unstable. The design of a stable inverse system is necessary in this situation. Two linear-time-invariant (LTI) systems with impulse responses $h[n]$ and $h_i[n]$ are inverse of each other if

$$h[n] * h_i[n] = d[n] \qquad (1)$$

where $*$ is the discrete-time convolution. Take the z-transform on both sides of (1), we have

$$\hat{x}(z) = h_i(z)h(z)x(z) = x(z)$$

or equivalently,

$$h(z)h_i(z) = 1 .$$

Therefore, the necessary and sufficient condition of design the inverse system is

$$h_i(z) = \frac{1}{h(z)} .$$

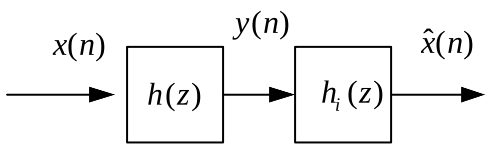

Fig 1. Cascade of a discrete-time system $h(z)$ with its inverse discrete-time system $h_i(z)$.

As an example, let $h(z) = 1 + 16.3z^{-1} + 67.4z^{-2} + 44z^{-3}$ be the transfer function of the transmitter. The zeros are -10, 5.5, 0.8, there are two zeros located outside the unit circle. The inverse system is

$$h_i(z) = \frac{1}{h(z)} = \frac{1}{1 + 16.3z^{-1} + 67.4z^{-2} + 44z^{-3}} .$$

It can be easily seen that $h_i(z)$ is unstable. If the input signal $x(t) = \sin(2p \cdot 5t)$, the reconstructed signal is shown in Fig. 2.

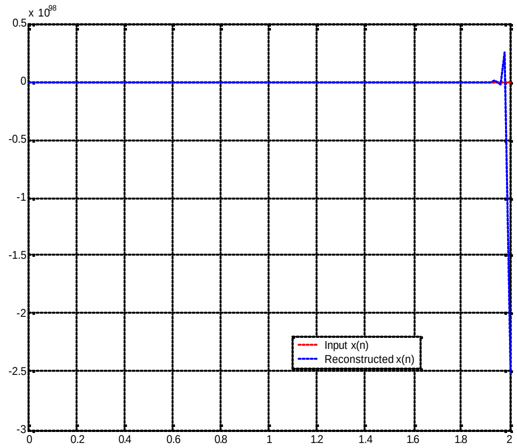

Fig 2. The signal $x(t)=\sin(2p\cdot 5t)$ and reconstructed signal by the inverse system

$$h_i(z)=\frac{1}{1+16.3z^{-1}+67.4z^{-2}+44z^{-3}}.$$

Problem description

If $h(z)$ is the system transfer function with poles lie outside the unit circle, the inverse system transfer function $h_i(z)=\frac{1}{h(z)}$ is unstable. For the inverse system to be stable and reconstructed the input signal , the scheme of the whole system is now modified as in Fig. 3. The research is to design a stable transfer function $h_i(z)$, with the criterion of least square error, that is to find $h_i(z)$ such that

$$\min J=\min\sum_{n=-\infty}^{\infty}\left[e(n)e^{*}(n)\right]=\frac{1}{2p\ j}tr\oint_{|z|=1}S_{ee}(z)\tfrac{dz}{z} \qquad (2)$$

where $e(n)=\hat{x}(n)-x(n-L)$ is the system error, the superscript * is the conjugate transpose operator and $S_{ee}(z)=\sum_{i=-\infty}^{\infty}R_{ee}(i)z^{-i}$ is the z-transform transform of $R_{ee}(i)=\sum_{n=-\infty}^{\infty}e(n)e(n+i)$.

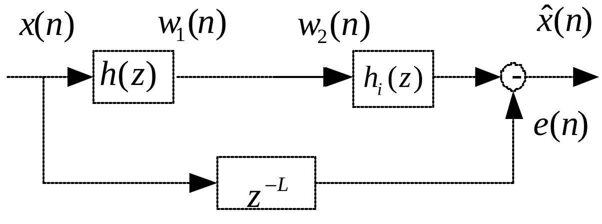

Fig 3. The proposed stable inverse system structure.

Optimal design of stable inverse system

The objective function of the optimal design is [3]

$$J = \frac{1}{2\pi j} tr \oint_{|z|=1} \left[cz^{-L} - h_i(z)h(z)\right]\left[cz^{-L} - h_i(z)h(z)\right]^* \frac{dz}{z}. \quad (3)$$

Suppose $h_o(z)$ is the optimal solution of $h_i(z)$ and let

$$h_i(z) = h_o(z) + e h(z) \quad (4)$$

where e is any real number, $h(z)$ is an arbitrary realizable stable rational transfer function with all poles lie in $|z|<1$. Substitute (4) into (3) and rearrange the equation in terms of e,

$$J = \frac{1}{2\pi j} tr \oint_{|z|=1} [J^{(0)}(z) + e J^{(1)}(z) + e^2 J^{(2)}(z)] \frac{1}{z} dz] \quad (5)$$

where

$$J^{(0)}(z) = c^2 z^{-L}(z^*)^{-L} - c(z^*)^{-L} h_o(z)h(z) - cz^{-L}h^*(z)h_o^*(z) + h_o(z)h(z)h^*(z)h_o^*(z)$$

$$J^{(1)}(z) = -c(z^*)^{-L}h(z)h(z) + h(z)h(z)h^*(z)h_o^*(z) - cz^{-L}h^*(z)h^*(z) + h^*(z)h_0(z)h(z)h^*(z)$$

$$J^{(2)}(z) = h(z)h(z)h^*(z)h^*(z)$$

To find the minimization of J, it is necessary that $\frac{\partial J(z)}{\partial e}|_{e=0} = 0$, or equivalently,

$$tr \oint_{|z|=1} J^{(1)}(z) \frac{dz}{z} = 0 \quad (6)$$

Now consider the spectral factorization of $h(z)h^*(z)$

$$\Delta(z)\Delta^*(z) = h(z)h^*(z) \quad (7)$$

where the zeros and poles of $\Delta(z)$ is inside the unit circle. Substitute (7) into (6), we obtain

$$tr \oint_{|z|=1} [h_o(z)\Delta(z)\Delta^*(z) - cz^{-L}h^*(z)]h^*(z) \frac{dz}{z} = 0 \quad (8)$$

It is known that the poles of $T(z) = h_o(z)\Delta(z)\Delta^*(z) - cz^{-L}h^*(z)$ lie outside the unit circle. $cz^{-L}h^*(z)\Delta^*(z)^{-1}$ can be written as

$$cz^{-L}h^*(z)\Delta^*(z)^{-1} = \left[cz^{-L}h^*(z)\Delta^*(z)^{-1}\right]_+ + \left[cz^{-L}h^*(z)\Delta^*(z)^{-1}\right]_- \quad (9)$$

where $[\bullet]_+$ and $[\bullet]_-$ denote poles lie inside the unit circle and outside the unit circle, respectively.

Hence

$$T(z)\Delta^*(z)^{-1} = h_o(z)\Delta(z) - cz^{-L}h^*(z)(\Delta^*(z))^{-1}$$
$$= h_o(z)\Delta(z) - \left[cz^{-L}h^*(z)(\Delta^*(z))^{-1}\right]_+ - \left[cz^{-L}h^*(z)(\Delta^*(z))^{-1}\right]_- \tag{10}$$

Substitute (10) into (9), we have

$$tr\oint_{|z|=1}\left\{h_o(z)\Delta(z) - \left[cz^{-L}h^*(z)\Delta^*(z)^{-1}\right]_+ - \left[cz^{-L}h^*(z)\Delta^*(z)^{-1}\right]_-\right\}\Delta^*(z)^{-1}h^*(z)\frac{dz}{z} = 0 \tag{11}$$

The poles of the third integrand in (11) lie outside the unit circle, by Cauchy's residue theorem [4]

$$tr\oint_{|z|=1} h_o(z)\Delta(z) - \left[cz^{-L}h^*(z)\Delta^*(z)^{-1}\right]_+\Delta^*(z)^{-1}h^*(z)\frac{dz}{z} = 0 \tag{12}$$

Now all poles of $\Delta^*(z)h^*(z)$ in (12) are located at $|z|>1$, by the calculation of variation, the sufficient condition for (12) to be true is the following

$$h_o(z)\Delta(z) - \left[cz^{-L}h^*(z)(\Delta^*(z))^{-1}\right]_+ = 0.$$

The optimal solution is

$$h_o(z) = \left[cz^{-L}h^*(z)\Delta^*(z)^{-1}\right]_+\Delta^{-1}(z) \tag{13}$$

The design procedure is summarized as

1. Obtain the spectral factor $\Delta(z)$ such that $\Delta(z)\Delta^*(z) = h(z)h^*(z)$。
2. The optimal stable inverse transfer function $h_o(z) = \left[cz^{-L}h^*(z)\Delta^*(z)^{-1}\right]_+\Delta^{-1}(z)$。

The characteristics of the proposed transfer function are

1. $h_o(z)$ is causal and stable.
2. If $h(z)$ is an infinite impulse response (IIR) filter, that is, $h(z) = \dfrac{b(z)}{a(z)}$, then

$$h_o(z) = \left[cz^{-L}\frac{b^*(z)}{a^*(z)}\Delta^*(z)^{-1}\right]_+\Delta^{-1}(z) \tag{14}$$

2. If $h(z)$ is a finite impulse response (FIR) filter, that is, $h(z) = b(z)$, then $a(z) \equiv 1$, and hence

$$h_o(z) = \left[cz^{-L}b^*(z)\Delta^*(z)^{-1}\right]_+\Delta^{-1}(z) \tag{15}$$

The spectral factorization of $h(z)$ [5]

If $h(z)h^*(z) = B_n z^{-n} + B_{n-1} z^{-(n-1)} + \cdots B_1 z^{-1} + B_0 + B_1 z + \cdots B_n z^n$, construct the matrix

$$\mathbf{T}_i = \begin{bmatrix} B_0 & B_1 & \cdots & B_i \\ B_1 & B_0 & \cdots & B_{i-1} \\ \vdots & \cdots & \ddots & \vdots \\ B_i & B_{i-1} & \cdots & B_0 \end{bmatrix}$$

with $B_i = 0$ if $i > n$. For i large enough, perform the Cholesky factorization so that $\mathbf{T}_i = \mathbf{D}^* \mathbf{D}$. The coefficients of the spectral factor $D(z) = d_0 + d_1 z^{-1} + d_2 z^{-2} + \cdots + d_n z^{-n}$ is obtained by the matrix

$$\mathbf{D} = \begin{bmatrix} * & * & \cdots & & & * \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & d_0 \\ \vdots & & & \ddots & & d_1 \\ \vdots & & & \ddots & & d_2 \\ * & * & \cdots & \cdots & \cdots & d_3 \end{bmatrix}.$$

An illustrated example for FIR system

If $h(z) = 1 + 16.3z^{-1} + 67.4z^{-2} + 44z^{-3}$, it is known that $h(z)$ is not minimum phase. The reconstructed signal by the direct reciprocal transfer function is shown in Fig. 2. By the proposed approach,

$$\Delta(z) = 0.8 + 13.4z^{-1} + 59.5z^{-2} + 55z^{-3} \text{ ,}$$

hence

$$h_o(z) = \frac{\bar{b}(z)}{\bar{a}(z)} \text{ ,}$$

where

$$\bar{b}(z) = (0.8z^3 + 1.2254546z^2 + 0.29636364z + 0.01818181)z^4$$

$$\bar{a}(z) = \left(55z^2 + 59.5z + 13.4\right)\left(z + 0.762147155\right)\left(z + 0.3196710268\right)$$
$$= 55z^4 + 119z^3 + 91.1682z^2 + 28.9927\,z + 3.2647$$

the roots of $\bar{a}(z)$ are 0.7621, 0.7621, 0.3197, 0.3197, all are located inside the unit circle. Therefore $h_o(z)$ is stable, the reconstructed signal is shown in Fig. 4(a). It can be seen that the envelope is almost the same as that of the input signal. With the system delay $L = 2$, the reconstructed signal is shown in Fig. 4(b).

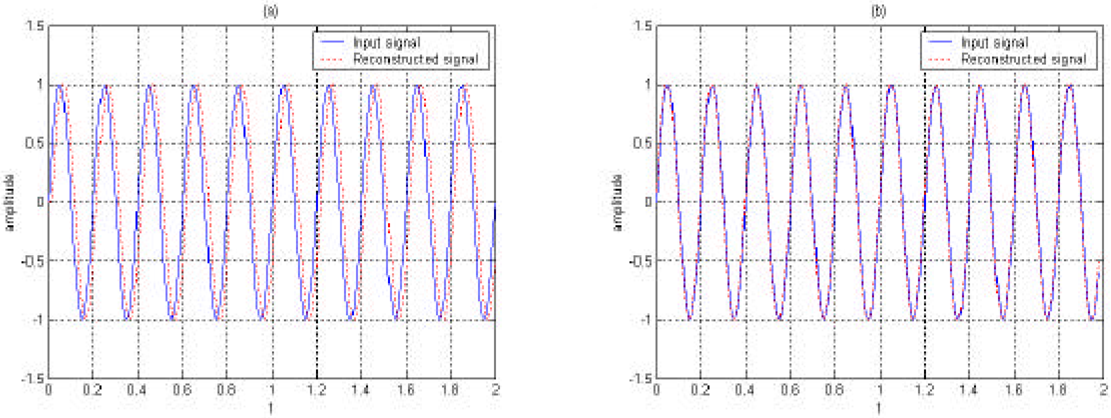

Fig 4. (a) The reconstructed signal by $h_o(z) = \dfrac{\bar{b}(z)}{\bar{a}(z)}$, (b) With $L = 2$, the reconstructed signal and the input signal.

An illustrated example for IIR system

If $h(z) = \dfrac{(1-2z^{-1})(1-3z^{-1})}{(1-.5z^{-1})(1-.25z^{-1})} = \dfrac{1-5z^{-1}+6z^{-2}}{1-.75z^{-1}+.125z^{-2}}$, the zeros are 2 and 3, hence the poles of $\dfrac{1}{h(z)}$ lie outside the unit circle. By the proposed design, the spectral factor of $1-5z^{-1}+6z^{-2}$ is $6.0000000001 - 4.999999999z^{-1} + .9999999998z^{-2}$ and the spectral factor of $1-.75z^{-1}+.125z^{-2}$ is $1-4z^{-1}+3.75z^{-2}$. Therefore

$$h_o(z) = \frac{\bar{b}(z)}{\bar{a}(z)},$$

where

$$\bar{b}(z) = 1 - 0.75z^{-1} + .125z^{-2}$$
$$\bar{a}(z) = 6.000000001 - 4.99999999z^{-1} + .999999998z^{-2}$$

the roots of $\bar{a}(z)$ are 0.5 and 0.333. It can be conclude at $h_o(z)$ is a stable system. The reconstructed signal is shown in Fig. 5(a). With the system delay $L=5$, the output is shown in Fig. 5(b). The both envelope is almost the same.

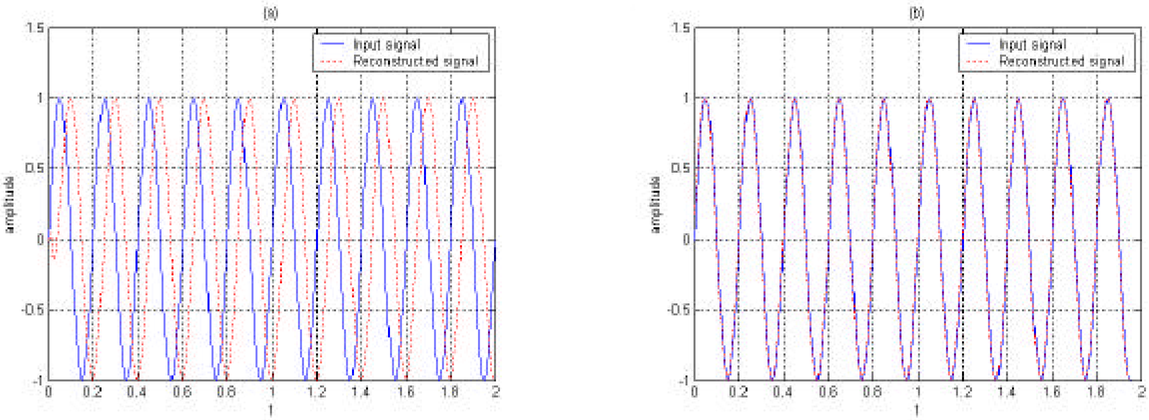

Fig 5. (a) The signal reconstructed by $h_o(z)=\dfrac{\bar{b}(z)}{\bar{a}(z)}$ (b) With system delay $L=5$, the reconstructed signal and the input signal.

Conclusion

The proposed inverse system is stable, with the system delay introduced, the inverse system can be causal. By the two examples, it can be seen that the stable inverse system can be obtained by proposed approach.

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