The Calculator and the Curriculum: The Case of Sequences and Series

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Abstract: Graphics calculators have the potential to influence the curriculum in several ways, including affecting what is taught, how it is taught and learned and how it is assessed. These relationships are exemplified for the particular case of sequences and series, which frequently appear in mathematics curricula near the end of secondary school and in the early undergraduate years. Attention will focus on the ways in which these mathematical objects can be represented, viewed, manipulated and understood by students using graphics calculators. Key concepts associated with sequences and series are examined from a calculator perspective. The paper provides an analysis of the mathematics curriculum through the lens of an available technology, with a view to providing suggestions and recommendations for both curriculum development and classroom practice.

The major significance of the personal technology of the graphics calculator is that it has the potential to be integrated into the mathematics curriculum, rather than be regarded as an 'extra' or as a 'teaching aid'. This paper provides an analysis of the relationships between the graphics calculator and one part of the mathematics curriculum, concerned with sequences and series, with a view to understanding the significance of the technology. The paper might thus be regarded as a companion to previous papers offering similar analyses, such as Kissane (1997) for probability, Kissane (1998a) for inferential statistics, Kissane (1998b) for calculus and Kissane (2002a) for equations.

To focus the analysis, it is convenient to use the structure suggested by Kissane (2002b), reflecting three different roles for technology in the curriculum. A calculator has a *computational* role, handling some aspects of mathematical computation previously handled in other ways. Secondly, a calculator has an *experiential* role, providing fresh opportunities for students to experience mathematics, and thus fresh opportunities for teachers to structure the learning programme. Finally, a calculator has an *influential* role, since the mathematics curriculum ought to be constructed with the available technology in mind; a curriculum constructed on the assumption that graphics calculators are routinely available might be expected to differ from a regular curriculum devoid of access to technology.

Throughout the paper, we use the Casio cfx-9850GB PLUS graphics calculator to illustrate the main connections between the mathematics and the technology. This calculator is widely used in senior secondary schools and the early undergraduate year, and does not have CAS capabilities. The choice of a non-CAS calculator is deliberate: at the present time, these are more accepted by curriculum authorities and also they provide substantial pedagogical support for students and teachers. Further, an analysis of the relationships between an algebraic calculator (ie with CAS) and the curriculum can easily be constructed using the present work as a basis.

Computational role

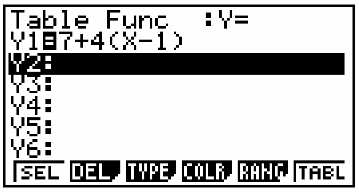
Sequences are important mathematical objects, perhaps best defined as functions with domain the set of natural numbers or a subset of these. Although sequences are generally infinite structures (as the domain is infinite), in practice we are frequently interested in a finite subset. Graphics calculators are of course finite machines and thus capable only of dealing directly with finite sequences. Indeed, in the case of school mathematics, most applications of sequences and series are concerned with finite examples, which have the most plausible practical significance for students.

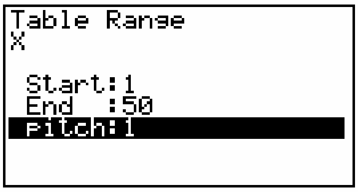
Generating a sequence

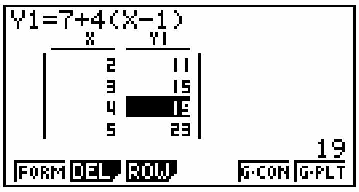
There are two essential ways in which sequences are defined, recursively and explicitly. A recursive definition specifies the relationship between successive terms of the sequence, as well as defining the starting point. An explicit definition provides a direct way of determining each term of the sequence. A sequence can be generated on a calculator in either of these ways.

Consider the elementary example of the arithmetic sequence, 7, 11, 15, 19, … . Successive terms of this sequence can be generated on a calculator by using the fundamental property that each term is four greater than the previous term, starting with a first term of 7. A graphics calculator allows this process to be automated, as shown in the screen below, in which successive terms after the second are generated by repeating the recursive command, *Ans* + 4, which involves only a single key press.

Although this can be an efficient way of finding a particular term, it may be quite tedious (and thus error-prone) for finding terms that are not close to the first term. An explicit formula for the same sequence is given by $T(n) = 7 + 4(n - 1)$. On a calculator, such a formula can be entered as a function and tabulated to produce successive terms efficiently, as shown below.





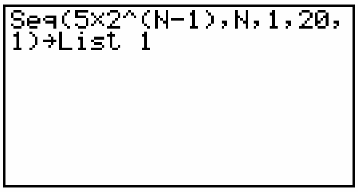


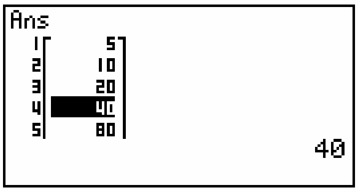
In this case, the commands above generate the first 50 terms of the sequence, substituting the calculator function Y1 = 7 + 4(X – 1) for the sequence function $T(n) = 7 + 4(n - 1)$.

In order to perform computations with sequences, it is necessary to first store them in the calculator in some way, which the procedures above do not accomplish directly. The essential means of doing this on a calculator is with an ordered *list*. In the case of the Casio cfx-9850GB PLUS, lists are restricted to 255 elements, which is more than ample for almost all secondary school purposes in practice. Sequences can be defined recursively or explicitly. To illustrate these alternatives, consider the geometric sequence with first term 5 and common ratio 2:

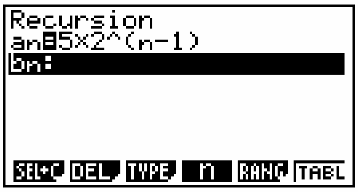
$$5, 10, 20, 40, \ldots$$

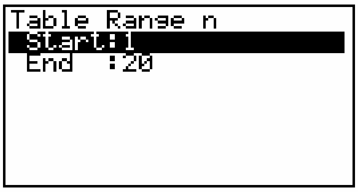
An explicit definition of this sequence $\{T(n)\}$ is $T(n) = 5 \times 2^{n-1}$, $n = 1, 2, 3, \ldots$ An explicit rule can be used in a calculator to generate successive terms. The screens below show how to generate the first twenty terms of this example and store them in List 1.

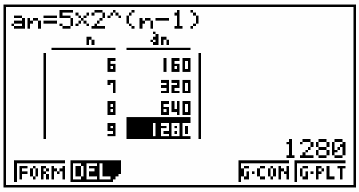




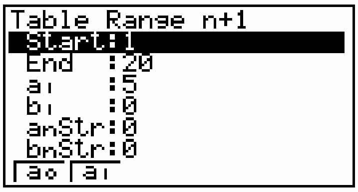
An explicit rule can also be used in the Recursion mode of the calculator (somewhat paradoxically):

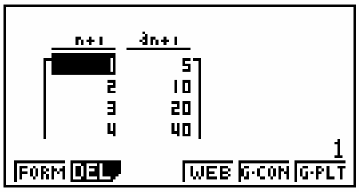




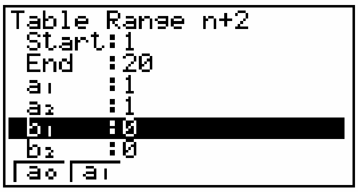


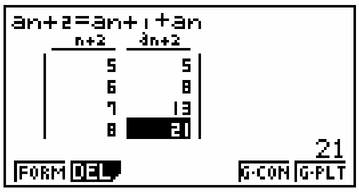
A recursive definition of the sequence is: $T(1) = 5$; $T(n + 1) = 2 \times T(n)$, $n = 1, 2, 3, \ldots$ This definition may be entered directly into the calculator and the sequence generated, as shown below.





Recursive definitions of sequences are not restricted to the relationship between successive terms; the screens below show the Fibonacci sequence, for which each term is the sum of the previous two terms, starting with the first and second terms being 1:

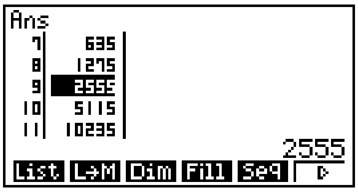


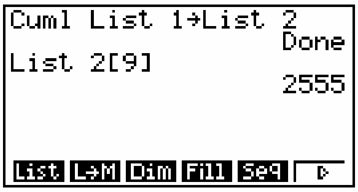


Once a sequence has been defined and stored in the calculator, the value of any term of the sequence can be readily determined by scrolling the relevant list.

Evaluating series

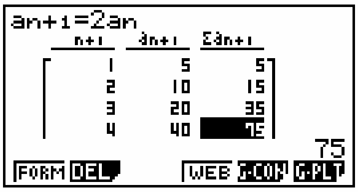
Adding successive terms of a sequence gives rise to a series, best regarded as the sequence of partial sums of a sequence. This is readily accommodated with a cumulate command on the calculator, one of the List functions available through the OPTN command in Run mode. The screens below show some examples of how this command produces the series corresponding to the geometric sequence above,





Similar computations are available for any sequence that has been stored as a list and may be performed directly in List mode or Statistics mode of the calculator as well as in Run mode.

In addition, the Recursion mode of this calculator allows for series to be determined routinely at the same time as a sequence is generated. Some results of this are shown below for the case of the geometric sequence defined previously:



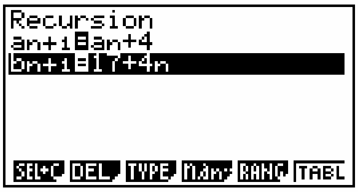
The screen shows the sequence of partial sums: 5, 15, 35, 75, … From this sequence, users can see that the sum of the first four terms of the sequence is 75.

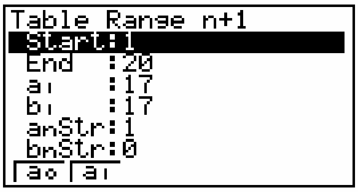
Alternatively, the calculator provides a summation function (Σ) to evaluate a series in Run mode, without the need to generate and store all the terms of the corresponding sequence. For some purposes, the storage limitation on the number of terms of a sequence prevents questions being addressed directly.

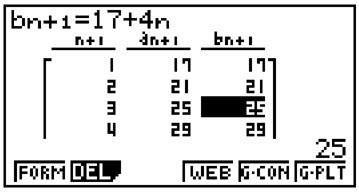
Experiential role

The defining aspect of an experiential role (Kissane, 2002b) is that the calculator provides students with opportunities not otherwise readily available for learning by experience. In this section of the paper, some examples of this are offered.

The ease of generating a sequence on a calculator offers students a chance to see the sequence as whole rather than merely focus on individual terms. Rather than use standard formulas for calculating a particular term or a series, students can investigate the sequence generated by a calculator. The ease of generation offers this sort of opportunity, which can be exploited in a number of ways. For example, the screens that follow show two different ways of generating the same sequence, one of them using a recursive rule (a_{n+1}) and the other using an explicit rule (b_{n+1}). Investigating these two rules simultaneously in this way seems likely to help students understand each one better than might otherwise be expected and to explore the conceptual links between them.



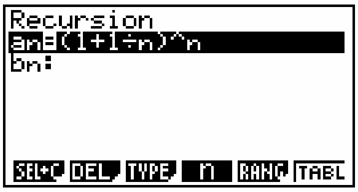


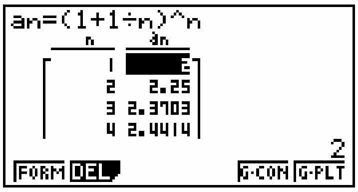


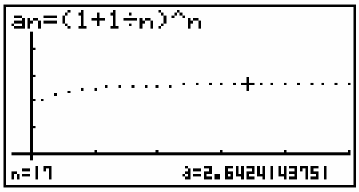
An important contribution of the calculator to student learning is that it offers ways of visualising sequences and series. Traditional approaches have tended to emphasise the numerical aspects, but adding a visual dimension offers an opportunity to understand better the concepts involved. This seems particularly the case for the concept of convergence.

Because a sequence can be regarded as a function with domain the natural numbers, a visual representation is essentially a scatter plot with the natural numbers on the horizontal axis and the terms of the sequence on the vertical axis. The screens below show a representation of the sequence

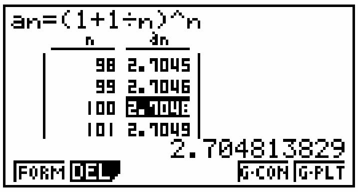
$$t(n) = \left(1 + \frac{1}{n}\right)^n, n = 1, 2, 3 \ldots$$

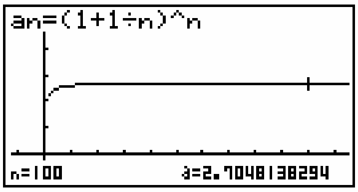




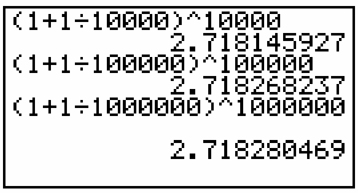


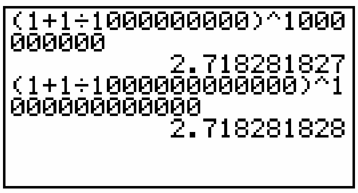
Despite the imperfections resulting from the chosen scales, the graph provides informal support for the idea that successive terms of the sequence are approaching a particular value. Students can explore this idea readily by graphing a larger number of terms, as shown below.





The screens above suggests that the sequence converges, with the graphical display reinforcing the impression conveyed by the numerical table of values. Of course, neither of these is a proof of convergence, but the role of the calculator is to offer conceptual support for the concept of convergence. The actual limit of the sequence can also be suggested by finding directly the values of terms with large values of *n*, as suggested by the screens below.

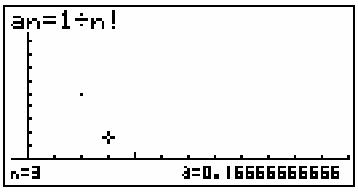


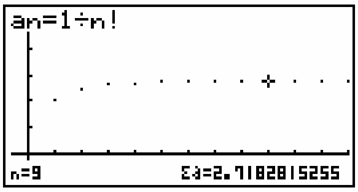


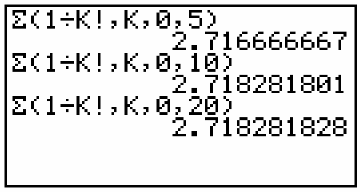
A convergent series is one for which the sequence of partial sums converges, and again a calculator provides helpful visual support for this idea. For example, the series given by

$$s(n) = \sum_{k=0}^{n} \frac{1}{k!}$$

can be readily seen visually to converge to *e*, with a very fast rate of convergence (compared with the previous sequence) as shown below. In this case, the parent sequence converges to zero, partly helping to make sense of the convergence of the series.





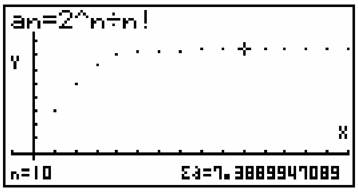


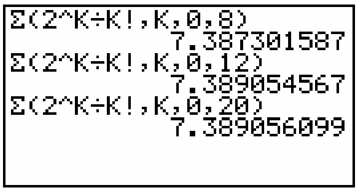
The third screen above shows again that results of this kind are also available with the Σ command without needing to obtain each of the terms of the sequence, as noted earlier. Students can get a sense of the rate of convergence by evaluating successive series directly.

Access to a graphics calculator allows for this sort of work to be readily extended to sequences for the exponential function. Thus, the series given by

$$s(n) = \sum_{k=0}^{n} \frac{2^k}{k!}$$

can be seen to also converge quite rapidly to $e^2 = 7.389056\ldots$, as the screens below indicate.





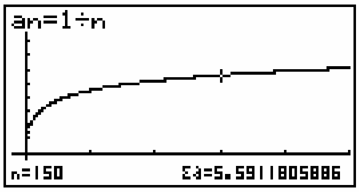
Generalisations of this kind are readily suggested by such explorations, so that students can investigate for themselves the consequences of replacing the 2 by another integer, or indeed by another real number, to get a numerical sense of the remarkable Taylor series result:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

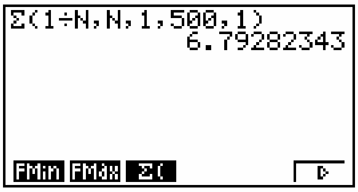
Explorations of this kind cannot be undertaken without access to a technology to quickly generate results, so that the calculator is providing an important mathematical experience not otherwise available. Of course, it does not always follow that sequences converging to zero have convergent series associated with them, and students can use a calculator to explore for themselves the harmonic sequence as a good illustration of this:

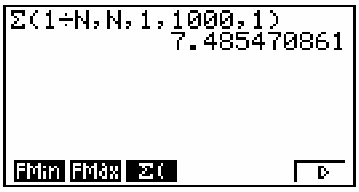
$$\frac{1}{1},\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},...$$

The screen on the right below shows the first 250 terms of the corresponding harmonic series, which visually suggests that convergence is not occurring, in contrast to the previous series.



The resolution on the screen has the potential to mislead students here, however, as it looks as if the graph is horizontal in places towards the right of the screen. A supplementary approach that confronts such a misconception is to use a summation command to evaluate many more terms of the series than the calculator is capable of graphing. The screens below show the sum of the first 500 terms and the sum of the first 1000 terms respectively of the harmonic series:





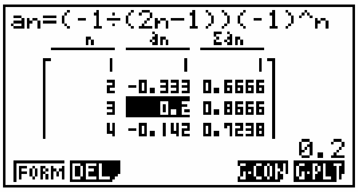
Although the calculator computes these sums by generating each term and adding them, it does not store the terms for later analysis, so the restriction on the maximum number (255) of terms of a sequence is not an impediment to the computation, and the lack of convergence is demonstrated a little more convincingly.

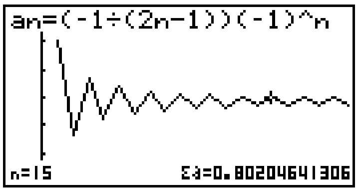
Although a formal mathematical proof is required to place such observations on a solid footing, the role of the calculator here is to help students make sense of the difficult ideas involved and perhaps also to provoke a search for a good analytic proof of results to support observations. Indeed, as ideas of convergence and divergence necessarily involve the infinite, no form of technology can do more than suggest what is happening, an important realisation for students to acquire and a powerful motivation for coming to terms with the formal mathematical arguments. In this case, the calculator provides a numerical and a graphical perspective, both important to supplement the symbolic perspective associated with the standard analytic proof of divergence.

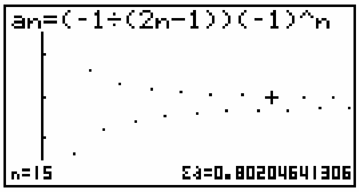
Explorations of other kinds of behaviour of sequences and series are available with these basic tools. For example, informal explorations of Gregory's series

$$\frac{\pi}{4}=\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+...$$

allow students to see the (slow) convergence of an oscillating series, as shown below:





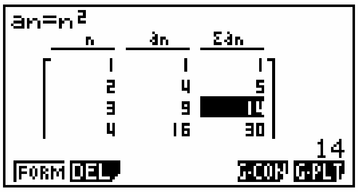


Successive terms of this series have been plotted above first with line segments joining successive points for visual effect, although a graph ought properly comprise only discrete points, as shown in the third graph above. Issues of this kind ought be adequately discussed in the classroom.

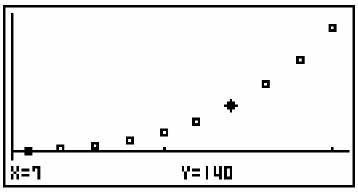
Another kind of learning opportunity offered by the calculator involves finding explicit formulas for series. A standard mathematics curriculum usually requires students to have some awareness of the (elegant) arguments for evaluating arithmetic and geometric series, but rarely involves other kinds of series, because of the complexities involved. Using a graphics calculator, some access to other ways of finding a formula for a series are available. To illustrate, consider the series

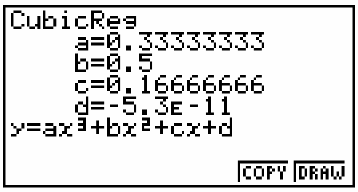
$$s(n) = 1^2 + 2^2 + 3^2 + ... + n^2$$

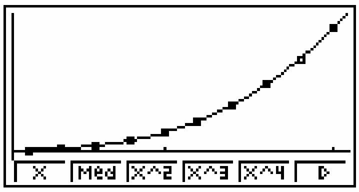
Successive partial sums of the series can be evaluated directly, as shown below, and the (finite) sequences involved can be readily transferred to the data analysis area of the calculator.



A scatter plot of the successive partial sums versus the number of terms shows a clear curvilinear relationship for the first ten terms as shown below.



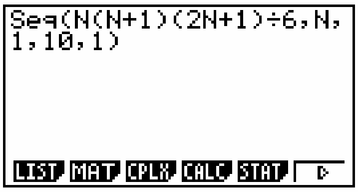


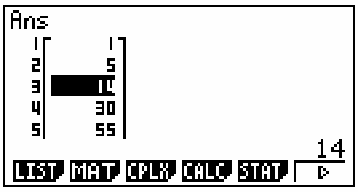


Within the limitations of the numerical accuracy of the calculator, students can see that a cubic relationship fits these data very well, matching nicely the pattern that might be expected on the basis of the sum of successive integers having a quadratic form. Provided they are prepared to accept the value of d above to be zero and the numerical coefficients to be decimal versions of fractions, students can use procedures of this kind to see that the series is determined by

$$s(n) = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} = \frac{n(n+1)(2n+1)}{6}$$

This result is readily verified on the calculator in various ways, one of which is shown below.





While such methods do not of course constitute a proof of the corresponding results, they offer students new opportunities to explore relationships of these kinds for themselves and provide some incentive to look for mathematical arguments that justify their observations. Cuoco (2003) recently described other ways of representing a table of results with an explicit formula, using successive finite differences, but space precludes a full treatment of this sort of technique here. Calculators readily permit numerical explorations of this kind, once sequences are stored adequately, and offer a stimulus to more advanced thinking about the empirical results obtained.

Influential role

As suggested in Kissane (2002b), a technology device such as a graphics calculator might be expected to influence opinions on which aspects of mathematics ought to be emphasised and regarded as important, provided the device is reasonably likely to be available to all students.

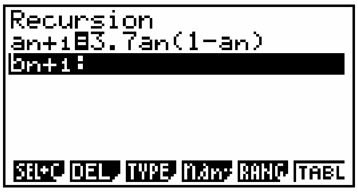
One clear implication in the present case is that the previous emphasis on computation of terms of sequences and series might reasonably be reduced, since students can readily find a given term of a given sequence and evaluate corresponding series directly on the calculator. It still seems important for students to appreciate the conceptually pleasing formulas for arithmetic and geometric sequences and the neat arguments provided to justify the evaluation of the corresponding series. The availability of the graphics calculator offers an opportunity to focus more attention on meanings and less on computations. In this vein, the conceptual links between recursive and explicit definitions of sequences deserve more attention than they have often received in the past.

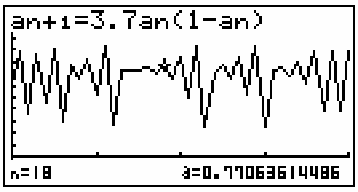
A second implication is that more attention might be devoted to the tasks of recognising sequences and finding ways to represent them that facilitate their evaluation. It is not an easy matter, for example, to represent the Gregory series above in a form that allows it to be examined, and students need help to think about sequences and series in this way. Similarly, explorations of the Taylor expansion of the sine function require students to represent the following series on a calculator:

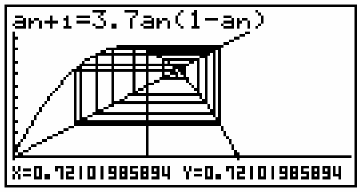
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$$

Less time spent on routine computations using standard formulas might free up some classroom time for such important learning. In addition, the curriculum might offer more opportunities to investigate approximations of this kind available through the use of sequences and series, since the approximations can be dealt with directly on the calculator, rather than relying on analytic aspects alone.

Finally, the graphics calculator offers students opportunities to explore in an intuitive way mathematical ideas that were previously inaccessible to them. A very good example of this involves elementary notions of chaos, described in more detail in Kissane (2003, p.152). The screens below indicate some of the possibilities, using a particular example of the logistic sequence. (Again, for visual effect, the discrete sequence has been represented here as if it were continuous.)







Activity of this kind has the additional advantage that it permits students to grapple with mathematical ideas that are seen to be current in popular literature, in stark contrast to much of school mathematics, which is centuries old. In an age when students are readily attracted to the new, and many of them are attracted away from mathematics itself, a conscious effort to move in this direction deserves some consideration.

Conclusion

A graphics calculator, such as Casio's cfx-9850GB PLUS offers new opportunities for students to learn about sequences and series, partly because it provides new ways of dealing with the computational demands involved and partly because it allows students to explore mathematics in several different ways at once. These two observations together suggest that a revised curriculum might be expected if technology of this kind is routinely available to students. Constructing such a curriculum is not an easy matter, demanding that we carefully preserve the best of traditional mathematics, and yet make some space for the advantages offered for new ways of looking at mathematics.

References

Cuoco, A. 2003. Match making: Fitting polynomials to tables. *Mathematics Teacher*, 96, 3, 178-183.

Kissane, B. 1997, The graphics calculator and the curriculum: The case of probability. In N. Scott & H. Hollingsworth (Eds), *Mathematics: Creating the future*, Melbourne, Australian Association of Mathematics Teachers, 397-404. [http://wwwstaff.murdoch.edu.au/~kissane/papers/AAMT97Prob.pdf]

Kissane, B. 1997, Chance and data: New opportunities provided by the graphics calculator, in W. C. Yang & Y. A. Hasan (Eds) *Computer Technology in Mathematical Research and Teaching* (pp 80-88), Penang, Malaysia, School of Mathematical Sciences.

Kissane, B. 1998, Inferential statistics and the graphics calculator, in W.C. Yang, K. Shirayanagi, S.-C. Chu & G. Fitz-Gerald(Eds) *Proceedings of the Third Asian Technology Conference in Mathematics* (pp 111-121), Tsukuba, Japan: Springer. [http://wwwstaff.murdoch.edu.au/~kissane/papers/tsukuba.pdf] (a)

Kissane, B. 1998, 'Personal technology and the calculus', *Reflections*, 23, 1, 28-31. [http://wwwstaff.murdoch.edu.au/~kissane/calculus/calculus.htm] (b)

Kissane, B. 2002, Equations and graphics calculators. In D. Edge & Y. B. Har (Eds) *Mathematics Education for a Knowledge-Based Era: Proceedings of Second East Asia Regional Conference on Mathematics Education and Ninth Southeast Asian Conference on Mathematics Education*, Selected Papers, Volume 2, (pp 401-408) Association of Mathematics Educators, Singapore. (a)

Kissane, B. 2002, Three roles for technology: Towards a humanistic renaissance in mathematics education. In A. Rogerson (Ed) *The Humanistic Renaissance in Mathematics Education: Proceedings of the International Conference* (pp 191-199), Palermo, Sicily: The Mathematics Education into the 21st Century Project. [http://wwwstaff.murdoch.edu.au/~kissane/papers/palermo.pdf] (b)

Kissane, B. (2003) *Mathematics with a Graphics Calculator: Casio cfx-9850GB PLUS*. Perth: Mathematical Association of Western Australia.