A Knowledge-Sensitive On-line Exercise for Developing Algebraic Calculation Strategies

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Abstract

Our observations indicated that students did not learn the calculation of algebraic rational expressions easily even if the students had enough skills to calculate numerical fractions. We suspected the reason is that algebraic calculations depend on several mathematical rewriting rules, which are usually hidden in black boxes at numerical calculations, and the students can not use the rules properly. The students can calculate numerical fractions automatically according to a few patterns they have learnt from experience and without thinking of the mechanism of the calculating steps. They can calculate even polynomials in such a way. They are surprised to find that their pattern-matching algorithms often fail in the calculation of rational expressions. Their patterns sometimes lead them to typical calculating errors lead by incorrect calculating rules, but they can not separate such kind of serious errors from other errors caused by careless manipulation.

Teaching rule-based calculation in a class was not enough for these students. They thought that the rules were too obvious to learn and they knew the rules already. But knowing the rules and using them properly are separate things.

Traditional exercises of calculations improved the students' manipulating skills and mostly decreased the errors caused by carelessness, but they did not improve the students' understanding on mathematical rewriting rules very much. The students, who could not separate the errors caused by incorrect rules from careless mistakes, were simply satisfied by the fewer but still existing ratio of the errors. They were repeating the errors lead by incorrect rewriting rules.

In this paper, the authors propose a new on-line exercise for such students to learn the calculation of algebraic rational expressions. In the exercise, a series of calculating steps from the question to the answer are displayed on a student's screen. Some of the steps may include calculating errors, and the others do not. The student is asked to examine

every calculating step, determine the type of the error for an erroneous step, and describe the rewriting rule of a correct step.
Such an exercise, when it was done on a pencil-and-paper basis, measured the students' sensitivities to calculating errors and their affinities to rule-based calculation.
The new exercise, if it is well organized with a manipulating exercise and an interactive instruction, is expected to help the students, who do not recognize the importance of mathematical rules in calculations, to develop the strategies themselves. We expect the system to help the students who used to improve very slowly in algebraic calculations even after a series of traditional calculating exercises.

1. Background

One of the most important purposes of learning algebraic calculation is to develop calculation strategies, in another words, to learn the use of rule-based algorithms, which some of the students have not learnt through their calculations with numbers. After the learning of calculating numbers, some students apply pattern-matching algorithms, which often lead them to calculating errors in algebraic calculations [1, 2]. The pattern-matching algorithms are fast in number calculations because they make calculations a series of automatic operations and hide the detailed mechanism in black boxes, but they hinder the students' learning of rule-based algorithms, which are the hidden mechanisms in the black boxes. The students who are good in using pattern-matching algorithms tend to think the rule-based algorithms too slow or as unnecessary detours in calculations.
As Buchberger [3] has warned of the danger of using black boxes *before* learning the mechanisms inside the boxes, we thought it necessary to develop an effective method to teach the mechanisms of algebraic calculation to high school students. The method must be individualized to each student, have quick and appropriate feedback, and have close relation with calculating exercise. An on-line system becomes a potential candidate to implement the method if we use technology to show the mechanism instead of hiding them. We have selected Web-based protocol and CAS as the key technologies for developing our system. The on-line individual exercise [4] and on-line step-by-step instruction [5] are some of the actual solutions of our system. In our experience of using the system, the most difficult students to teach are those who apply pattern-matching algorithms in algebraic calculations and do not recognize the need of rule-based calculation.
The students' activity called self-explanation in [6] could be a help to overcome this obstacle of our students.

2. Manipulating Skills and Knowledge of Rule-based Calculation

After observing *curious mistakes* in our students' answers using both pencil-and-paper exercises and interactive on-line exercises, we came to the hypothesis that the calculation of symbolic fractions requires a student to possess two different abilities, manipulating skills and active knowledge of rule-based calculation. The lack of the active knowledge part is the current issue of our students in trouble. "Active" knowledge, in this paper, is meant the knowledge that we use fluently. Traditional exercises with pencil-and-paper are not effective to learn the knowledge of rule-based calculation or to activate the knowledge although they always improve manipulating skills and in some cases refresh the knowledge of rule-based calculation. The students with little knowledge often go through the exercise using pattern-matching algorithms without learning the rule-based calculation.

To obtain a clearer view to the issue, we tried to measure the two students' abilities independently. We prepared two types of tests, and gave them to our 45 students (age 15) after three units of lectures on algebraic calculations of polynomials and rational expressions.

One of them was a traditional calculation test of rational expressions, which mainly examined their manipulating skills. We were also able to find the errors, especially the one lead by incorrect rewriting rules, among the calculating steps written in the answer sheets. We call the test a "skill-sensitive" test.

(1) An example of correct calculations

$$\frac{\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}}{\dfrac{a+x}{a-x}+\dfrac{a-x}{a+x}}=\frac{\dfrac{(a+x)^2-(a-x)^2}{a^2-x^2}}{\dfrac{(a+x)^2+(a-x)^2}{a^2-x^2}}=\frac{(a+x)^2-(a-x)^2}{(a+x)^2+(a-x)^2}=\frac{(a^2+2ax+x^2)-(a^2-2ax+x^2)}{(a^2+2ax+x^2)+(a^2-2ax+x^2)}=\frac{4ax}{2a^2+2x^2}=\frac{2ax}{a^2+x^2}$$

(2) An example of calculations with errors based on incorrect rewriting rules

$$\frac{1}{x(x+1)}+\frac{1}{(x+1)(x+2)}+\frac{1}{(x+2)(x+3)}=\frac{(x+2)(x+3)+x(x+3)+x(x+1)}{x(x+1)(x+2)(x+3)}=\frac{3x^2+9x+6}{x(x+1)(x+2)(x+3)}=\frac{3x(x+3)+6}{x(x+1)(x+2)(x+3)}\neq\frac{3(x+2)}{x(x+1)(x+2)}=\frac{3}{x(x+1)}$$

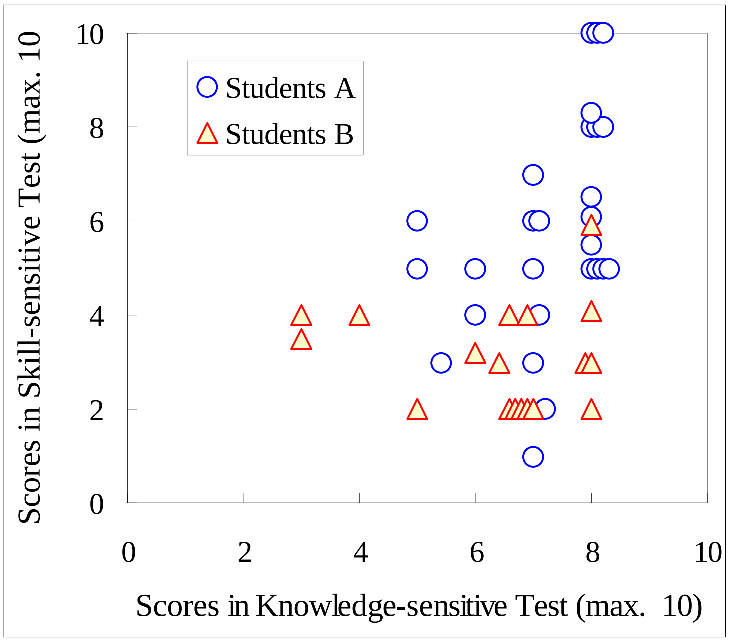
(3) An example of calculations with careless mistakes

$$\frac{x}{x-\dfrac{x+2}{x+2-\dfrac{x-1}{x}}}\neq\frac{x}{x-\dfrac{x+2}{\dfrac{x^2+2x-x-1}{x}}}=\frac{x}{x-\dfrac{x^2+2x}{x^2+2x-x-1}}=\frac{x}{x-\dfrac{x^2+2x}{x^2+x-1}}=\frac{x\left(x^2+x-1\right)}{x^3+x^2-x-x^2-2x}=\frac{x^2+x-1}{x^2+x-1-x-2}=\frac{x^2+x-1}{x^2-3}$$

Figure 1 Examples of the calculations at the knowledge-sensitive test

The other test requested the student to evaluate the already written ten calculations, some of which are shown in Figure 1. It is designed to confirm their knowledge of rewriting rules. In the ten calculations, four had errors caused by incorrect rewriting rules, three had careless mistakes, and three were correct calculations. They are listed in random order. We asked the students to point to the exact location of the errors if they exited, and to describe the possible reasons or types of the errors in the testsheet. Because all the calculating steps were already written on the sheet, it was an easier problem for the students with the knowledge of rule-based calculation but rather difficult problem for the students without it. We call the test a “knowledge-sensitive” test.

For our students, correct calculations were the easiest to evaluate (93% were right evaluations), and the calculations with careless mistakes were the most difficult ones (53% were right evaluations). The scores in the knowledge-sensitive test were largely affected by the evaluation of incorrect calculations caused by incorrect rewriting rules (71% were right evaluations).



Students A:
made no errors caused by incorrect rewriting rules in skill-sensitive tests.

Students B:
made calculating errors based on incorrect rewriting rules in skill-sensitive tests.

Figure 2 Score distribution of the knowledge-sensitive and skill-sensitive tests

Figure 2 shows the score distribution in the two tests. The horizontal axis shows the score of the knowledge-sensitive test, and the vertical axis shows the score of the skill-sensitive test. Each marker on the two-dimensional field is related to a student’s scores on both tests. Students A are the students who didn’t make any errors caused by incorrect rewriting rules in their own calculations at a series of skill-sensitive tests, and Students B are the students who did make the errors based on incorrect rewriting rules in the tests. Obviously, the students whom we have stronger concern are students B.

A feature of the data distribution in Figure 2 is that there are no data in the upper left field, so the data are divided roughly into three fields. The students with high score in the skill-sensitive test occupy the upper (right) field. There are only students A in this field, and they don't need any additional teaching described in this paper.

The students who have low score in the knowledge-sensitive test occupy the (lower) left field. There are only students B in this field, and they are the students who need the teaching of rule-based calculation. The knowledge-sensitive exercises proposed in this paper aim to help them.

The lower right field in Figure 2 is occupied by the students with high score in the knowledge-sensitive test but low score in the skill-sensitive test. There are two types of students, students A and students B, in this field, and it is difficult to separate them only with the result of the two tests. We suspect the students A in this field only need calculating exercises to improve their manipulation, but the students B need additional help to activate their knowledge of rule-based calculation when they perform their calculations. They will be helped to refresh their knowledge by the current system with the on-line exercise, where it detects the errors caused by typical incorrect rewriting rules, combined with the step-by-step instruction. The knowledge-sensitive exercise described in the next sections may also help them.

3. Method to Teach an Understanding of Rule-based Calculation

We need to develop a subtle method to teach rule-based calculation to the students, who lack the knowledge but don't recognize it by themselves. The method must find the students in trouble, let them know the necessity of rule-based calculation and teach the rules effectively.

Finding the students in trouble is the first step of the method, and we propose to use the knowledge-sensitive test for this purpose. The skill-sensitive test is not appropriate to find them because lower skills do not necessarily accompany less knowledge of rule-based calculation according to Figure 2.

Convincing the students of their problem and letting them recognize the importance of rule-based calculation are the second and most important step of the method. They are not easy because the students in trouble rarely recognize the problems by themselves. They can neither find errors in their calculation sheets nor categorize the errors according to the possible reason by themselves. Adding to that, they usually have some confidence in their calculation because of their success in numerical fractions and polynomials, where calculation can be done by pattern-matching method. Those are the reasons why a traditional lecture of rule-based calculation in a class is not enough.

We propose self-explanation [6] instead of calculation as the students' activity for this purpose. In the self-explanation, students are to explain the mechanism of every calculating step. If a step is erroneous, they explain the possible reason of the error. If the step is correct, they explain the rewriting rule applied at the step. In this activity, pattern-matching algorithms hardly help the students, and they have to find erroneous calculating steps and/or explain the mechanism of correct and incorrect calculating steps by themselves. Naturally, their poor knowledge of rule-based calculation becomes apparent to themselves. Quick feedback to misjudges or incorrect explanations tell the students their problems and let them recognize the need of learning rule-based calculation. We can use numerical counter examples to convince the students of their misjudgment [2], and also explain the possible cause of the error or the rewriting rule applied to the calculating step.

Teaching the rule-based calculation is the last step of the method. The step-by-step instruction in the current system [5] has developed for this purpose, and the knowledge-sensitive activity like self-explanation also boosts the learning. The knowledge-sensitive exercise is supposed to be effective in all three steps of the teaching method of rule-based calculation.

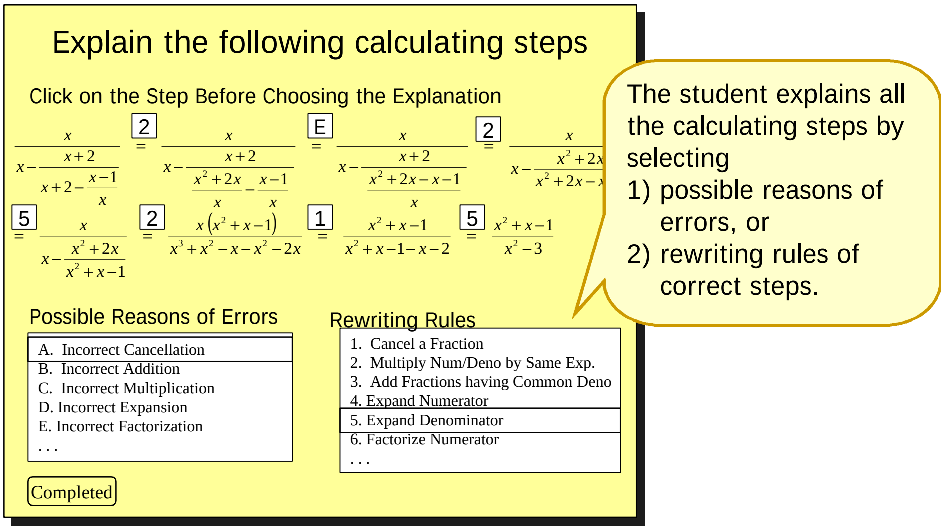


Figure 3 On-line Knowledge-sensitive Exercise

4. Knowledge-sensitive On-line Exercises

We have implemented a knowledge-sensitive exercise as shown in Figure 3 into our on-line system. In the exercise, every screen has a series of calculating steps of a rational expression starting from the question to the answer. In the calculating steps, none, one or more steps may include calculating errors. The student is requested to judge if there are any errors included in the steps. If he/she finds an error, he/she selects the possible reason of the error from the pop-up menu in the lower left corner of the screen. If he/she finds plural errors in the calculation, he/she repeats the process the necessary number of times to announce all the errors he/she has found. Then, the student is requested to explain the mechanism of the other correct calculating steps by selecting the rewriting rules applied to the steps from the menu in the lower right corner.

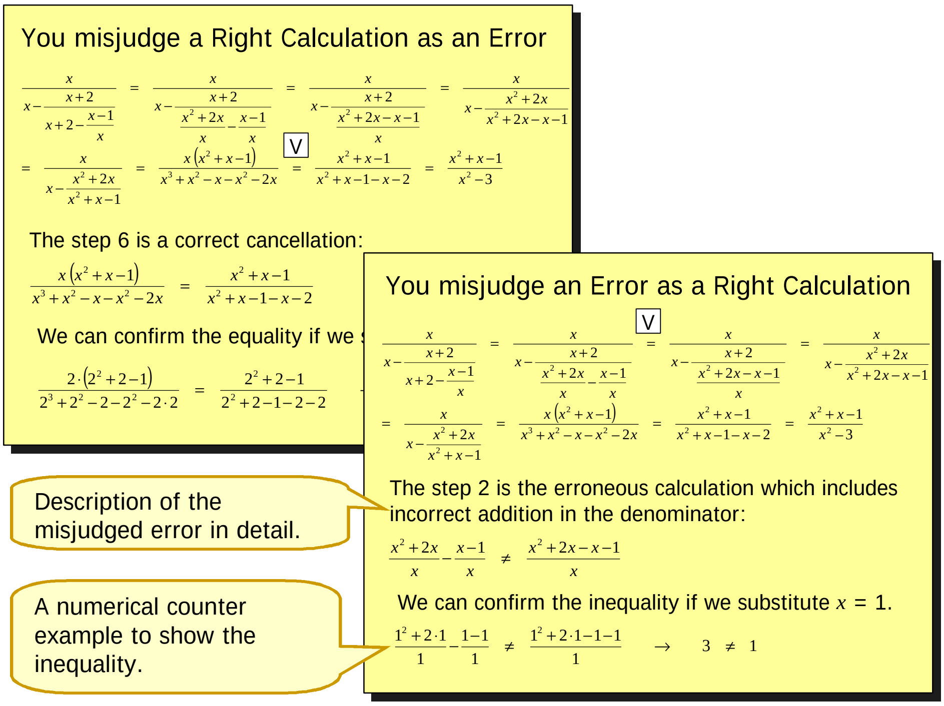


Figure 4 Example feedbacks to the misjudgments

From a student's misjudgment, the system knows what kind of a rewriting rule he/she does not understand. Figure 4 shows example feedbacks to students' misjudgments. If a student misjudged an erroneous calculating step as the right one and explained the rewriting rule as "Add fractions having common denominators", the feedback will be the description of an incorrect rewriting rule applied to the step. Adding to the description, a numerical counter example is displayed to confirm the inequality of two expressions before and after the calculating step. Every misjudgment receives the feedback in the form of similar description and a numerical example or a numerical counter example.

5. Discussion

We have discussed the importance of rule-based calculation in the algebraic rational expressions as opposed to the pattern-matching algorithm in algebraic calculation. As a result, our teaching method uses technologies to show the mechanism inside the automatic black box but not to bypass the complicated calculations for conceptual understanding or solving real life problems. We require the learning of the mechanism because we have seen many students who have avoided learning the mechanism and later complain of the lifelessness of mathematics.

We also would like to stress the importance of learning the relationship between algebraic expressions and descriptions in a natural language. For deeper understanding of a mathematical expression, we should learn to convert them in both directions, i.e., describing the mathematical expression in sentences of a natural language and expressing the content of sentences written with a natural language in the mathematical expression. The knowledge-sensitive exercise seeks the former direction and the step-by-step instruction meant to direct the latter one.

6. Conclusion

Students have to learn both manipulating skills and rule-based knowledge to calculate algebraic rational expressions successfully. There were students who showed their lack of rule-based calculation even after certain amount of calculating exercises. A knowledge-sensitive test could find the students in the most serious trouble.

We propose to introduce a knowledge-sensitive exercise, adding to existing interactive exercises and step-by-step instruction. The integrated system is expected to help students in trouble to improve competency in using rule-based algorithms in algebraic calculations.

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