Visualising functions of two variables with Cabri II Plus

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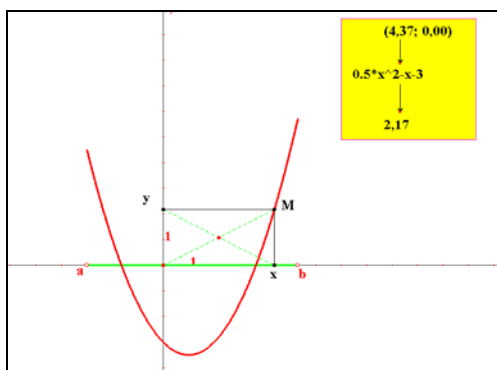(Institute of Research for the Teaching of Mathematics in Toulouse)

Abstract

Thanks to the new tools of Cabri II Plus, I will show how to represent such surfaces in different perspectives using very basic knowledge about functions of one variable. I will also show a powerful model for representing tangent planes to such surfaces.

Using colours for the sections of these surfaces with respect to the xOz and yOz planes and with the ability of modifying the coefficient of the perspective it will permit us to get a wonderful visualisation of these functions

1. Different ways to visualize curves of functions of one variable

1.1. Curve of a function as a locus of points

To obtain the curve of the function f defined with $f(x) = 0.5*x^2-x-3$, we have :
 1/ written on the screen of Cabri the expression $0.5*x^2-x-3$ using the new tool of Cabri II Plus, « **Expression** »
2/ Built one generic point M of this curve having (x ; f(x)) as coordinates (x is the abscissa of a point chosen on a segment [a ;b] included on the x axis; f(x) is calculated in using the new tool of Cabri II Plus « Evaluate an Expression »)
The curve of f is obtained as the locus of this generic point when the point "x" moves on its segment.

This construction made by students will give meaning to the operational register of function. Indeed it will become a routine : I prefer saying that we have managed a good "scheme" of using Cabri for constructing curves of functions. My students use to approach curves of functions in this way

1.2. Recording this construction to get a macro

We have recorded this construction (that can be used now as a new tool of Cabri II Plus)with:
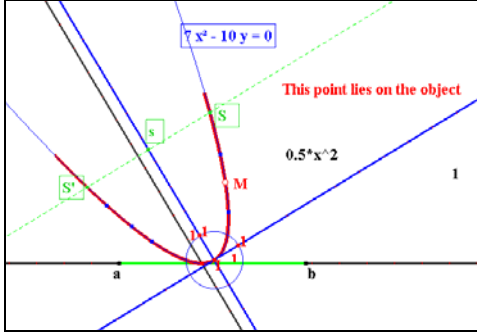The expression, the segment on the x axis and the axis **as initial objects**
The curve as a **final object**
This macro is the first step leading to a decontextualized understanding of the notion of curve of functions. Students built their own tools for progressing in math knowledges. It is a new starting point for the next stage in the direction of other close knowledge

1.3. Examples of the using of this macro

1.3.1. An experiment about parabolas

We will use this macro to construct the curve of the $0.5*x^2$ function in the following black system of axes. The problem that can appear is : is this curve the curve of a $m*x^2$ function in an orthogonal system of axis? It comes from the observation that the obtained curve looks like a parabola having as an orthogonal symmetry axis, a parallel line to the y black axis

Experiment :

♦ S is a point lying on the obtained curve, S' is the second intersection point of a line passing through S and perpendicular to the y black axis? s is the midpoint of the segment [SS'].
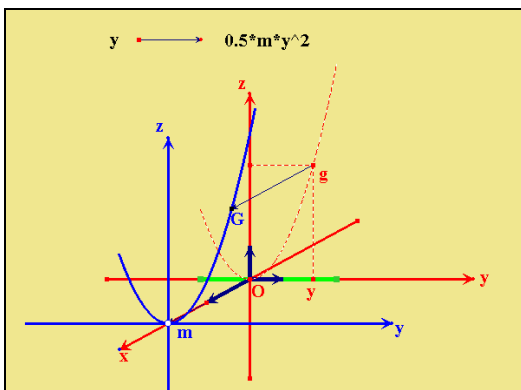
♦ If we drag point S on our curve, the trace of point s gives a parallel line to this y black axis. So we can **conjecture** that the perpendicular bissector of one particular segment [SS'] is a symetry axis of the curve.

♦ **Verification**: we built the orthogonal system of axis having this perpendicular bisector as a y blue axis and a perpendicular line to this one passing through I (I is the intersection point between this perpendicicar bisector and our curve). So if our conjecture is right, the right locus will be a parabola for Cabri and it will be able to confirm that the equation is also like y = m*x². M is a point chosen on our curve. The blue curve is a parabola passing through 5 points of our curve. If we test the property "Is point M a point belonging to this parabola?", the answer is yes even we drag M on the curve. The confirmation for the equation is also obtained.

This experimental way of practising math can be used by the teacher to show to his students, in the classroom, the power of exploration and investigation of such a tool (this specific dynamic geometry software). It contains like in experimental sciences a stage of construction called the protocol but this protocol is very special because it has consequences about that could be experimented (a point can be dragged or not for example a locus can be drawn or not). It contains also conjectures that can be validate with the eyes, but dynamically. These validations led with constructions are a modelisation of a reasoning with analysis and synthesis in the Cabri microworld

1.3.2. An example with parameters

This example will allow the teacher to work with his the students, the special link between the operational level of function and the graphic level.
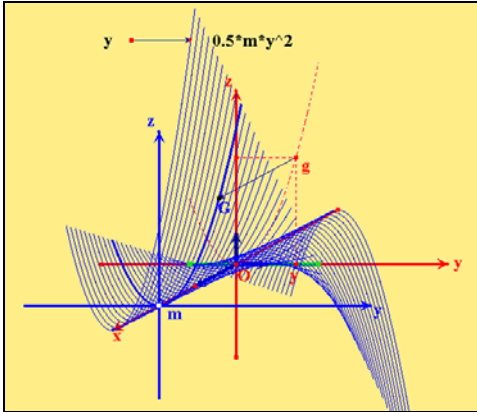

We have first constructed the curve (which is in red and obtained as the locus of g) of the function z = 0.5*m*y² in the system of axis yOz with the value of m being the abscissa of point m on the x abscissa.

We have translated this curve to put it in the plane of the space x = m (in fact we have translated g in G and ask the locus of G when y moves on the green segment).

In the classroom, in France, I use to guide my students in the information lab, to built step by step such a file (2 students in front on one computer: one manipulating and another writing the report of all what is done). The aim of such an activity is to become confident with the software and to conjecture results that can be observed only with this tool and not in a paper and pencil activity and also to visualise objects students will never have time to built even they would want to do it. The
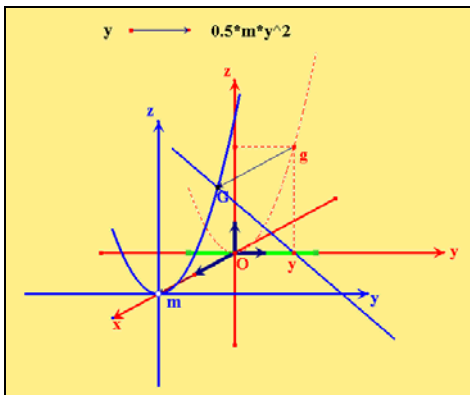
teacher is not stopped by the high mathematic level of the subject. He can use complex objects to work on basic concepts in both the algebraic field (work on the parameter) and the geometric field (visualise and conjecture)
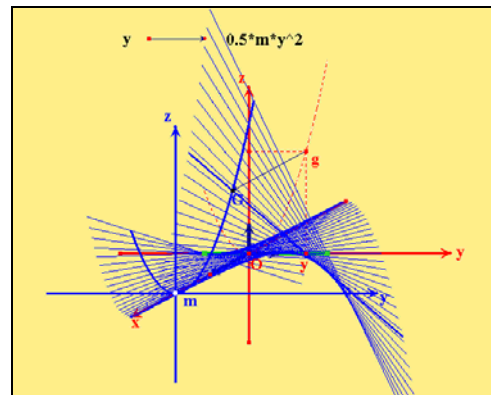

We obtain this surface when we ask the locus of our curve when point M moves.

So, one important using of parameters can be to introduce space figures starting from curves of functions.

It easy to understand that this example shows the continuous link established between very different subjects in a math curriculum. In France we did not use to teach surfaces in highschools in relation with curves of functions of one variable until some teachers began to use Excel. The use of Cabri I have discribed is a realistic practice because students can built these files and understand this link in manipulating
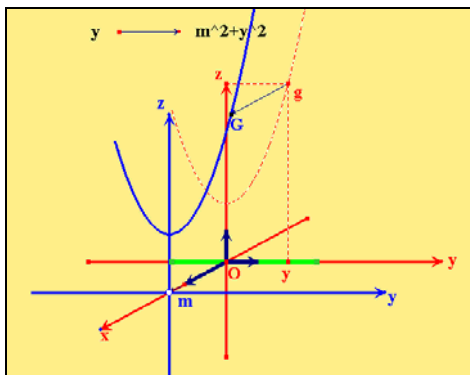

Here the blue segment has been got as the locus of G when m moves.

Here we have asked for the locus of this locus when y moves to get the same surface with other sections.

Remark: we have not used in this example the previous macro because m has not been put as an initial object when we have created this macro.
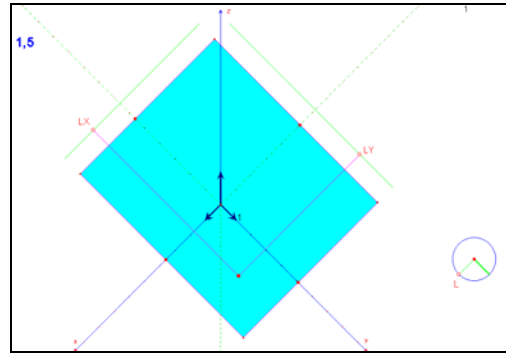
Below we have only change the expression and the file has been upgraded immediately.

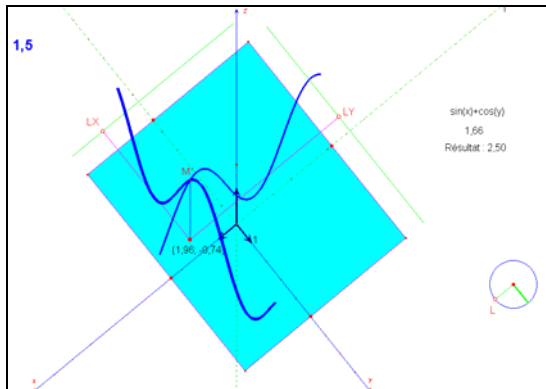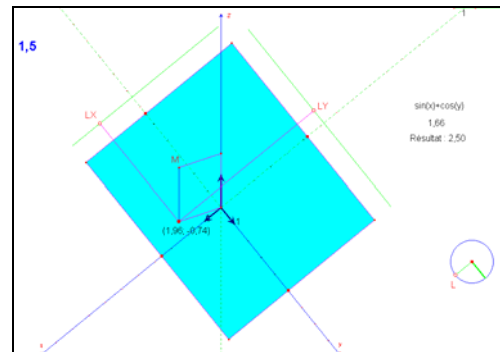## 2. Surfaces in a military perspective

We use a representation of space in a parallel perspective where the vanishing line is a vertical line and the floor plane parallel to the plane of the observer's eyes : It is an axonometry called in French: "military perspective". The coefficient of the perspective permits us to simulate the position of the observer to increase or decrease the perception of the shapes of the surface.

The blue rectangle in the left-side is the one where a function of two variables x and y will be defined.

We define a function f with an "Expression"; here, f(x ; y) = sin(x) + cos(x).

We calculate the value z of this function for the coordinates of a point of the blue rectangle (this point is led with the two points LX and LY created on special segments). The point M of the surface is plotted with measurement transfer of k*z (k is the coefficent of the perspective) on Oz.
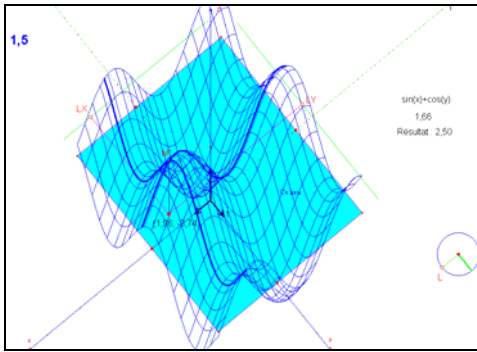
The locus (L1) of point M when LY moves is the thick blue curve which is the section between the surface z = f(x ; y) and a plane parallel to the yOz plane.

The locus (L2) of point M when LX moves is the second blue curve which is the section between the surface z = f(x ; y) and a plane parallel to the xOz plane.
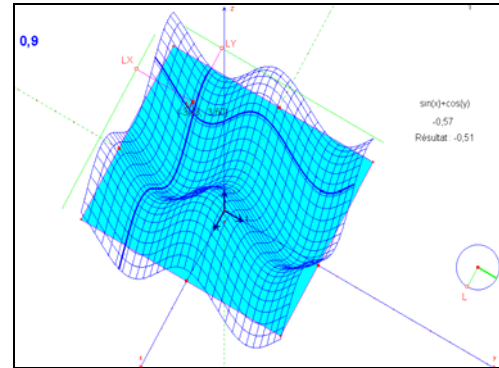
(To get the representation of the surface we have first chosen in the preferences of Cabri in the Loci Options**Locus of points: do not link points; Locus of lines or circles: do no ask for envelope.)**

Such files have been shown in teachers training to the use of Cabri to teach space representations. These teachers have appreciate to be tought such knowledge because these constructions need themto have math knowledges and permit them to create new and true math problems including an experimental way of practising math for their students. They convinced them that their students will not be better than them because the tool used is a computer. They can understand the difference between an artefact and an instrument (an instrument is an artefact with the schemes of using this artefact).
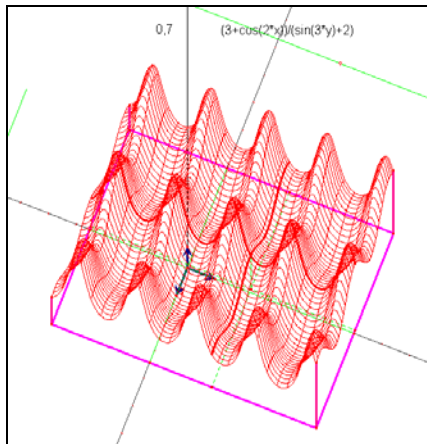
The representation of the surface realised on the lefthand side, is obtained in asking Cabri for :
The locus of (L1) when LX moves.
The locus of (L2) when LY moves.
It is possible to change the point of view in dragging point L and so the surface turns around the z axis.

Here we have changed the coefficient of the perspective and increased the number of curves to get this beautiful representation.
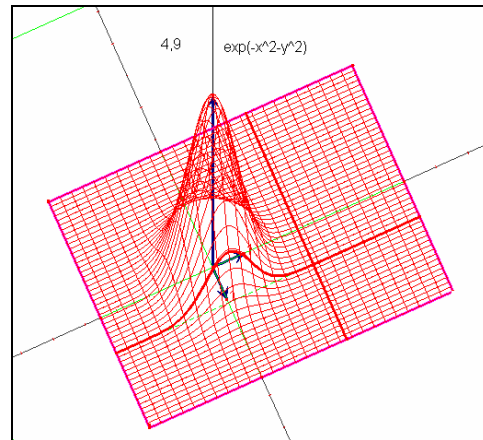

To obtain a gallery of surfaces

If we change only the formula in our "Expression", the surface will be changed very quickly. So it is very easy to create surfaces. Below, you have some of the ones I have got.
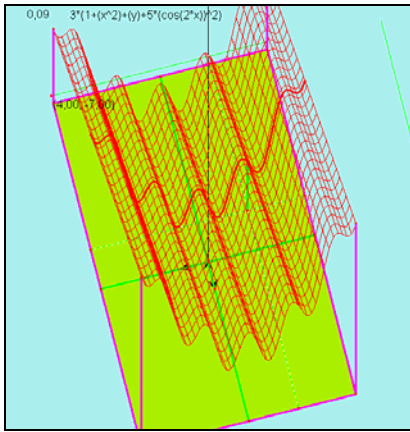

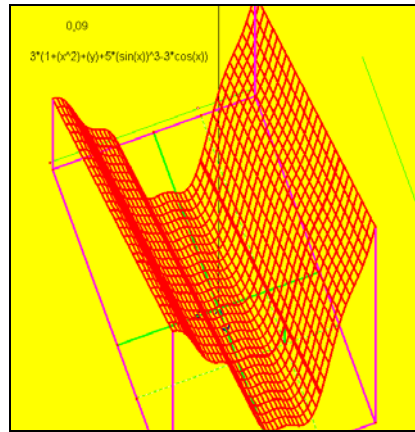$f(x ; y) = (3+\cos(2x)/(\sin(3y)+2)$

$f(x ; y) = \exp(-x^2-y^2)$

Apart the esthetic pleasure felt by teachers or students in these symbolic games, we involved them to explore to investigate, to be astonished and at last to conjecture. We obtain the devolution of a problem which is their own problem, their private problem in a first stage. In a second stage, we will obtain from teachers or students the devolution to the sharing of their researches so we will reach the rich stage of the scientific debate (in the meaning of Marc Legrand)
They will build in front of a special private task a path leading to technics hiding the theory the teacher will teach them because they will need
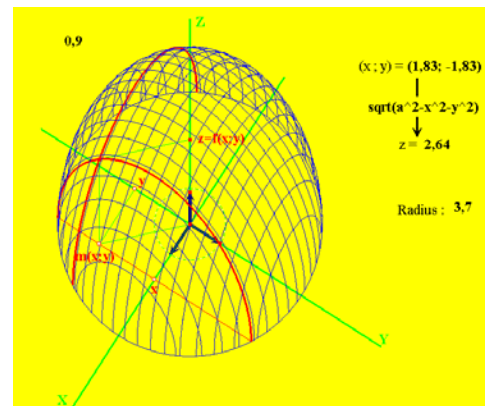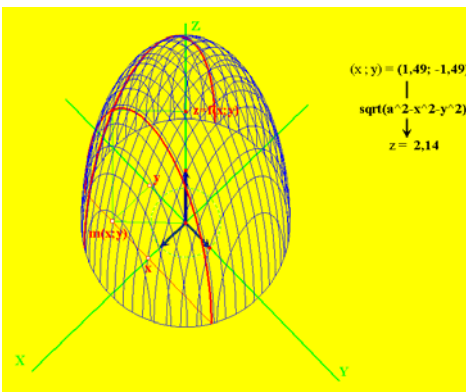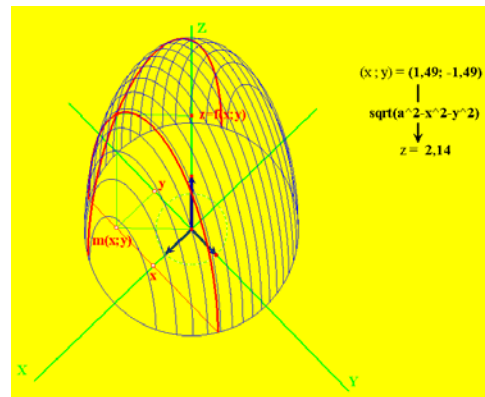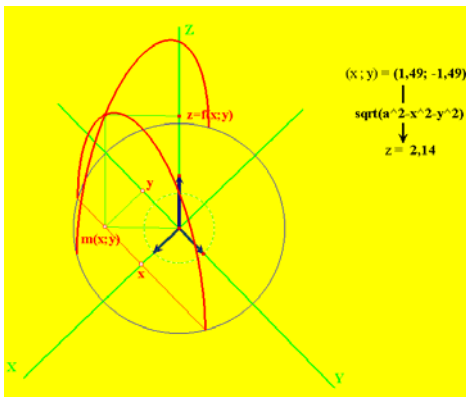
f(x ; y) = 3(1+x²+y+5cos²(2x))   f(x ; y) = 3(1+x²+y+5sin²(2x))
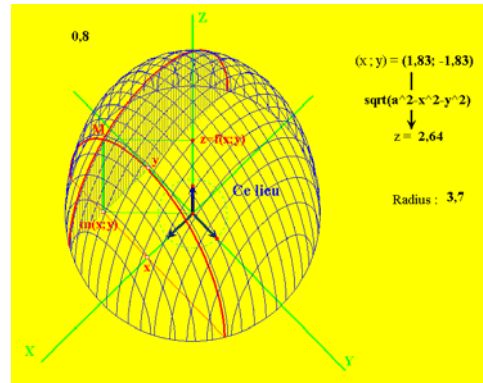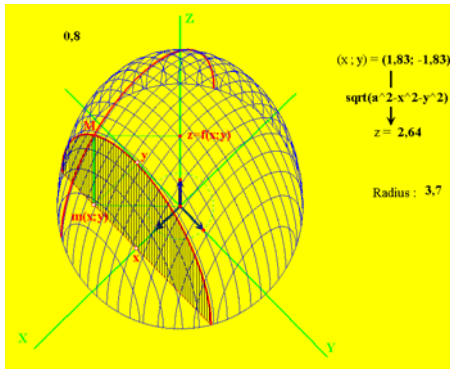
3. Representing a sphere as a surface z = $\sqrt{r^2 - x^2 - y^2}$

We can use the same file to get the constructions shown below

This could be understood as the shadow of half a sphere put on the sand of a beach and lightened with the sun of the end of the afternoon

We have changed the coefficient of the perspective in order to get a shadow looking like what we expect to get!

We have constucted the vertical segment [mM]We have obtained the section of this part of sphere with a vertical plane as the locus of this segment when y moves.
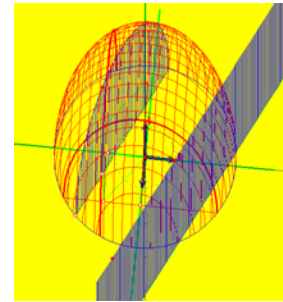
Same thing when x moves

Behind this work, there are only the Pythagor and circle notions that can be explored only in this information field. Sphere are very complex objects to be represented and visualise even in Math books
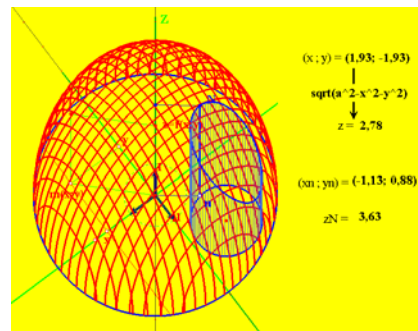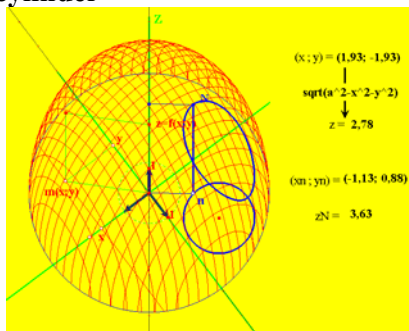
4. Intersections between a sphere and
4.1. Vertical planes

Here we have visualise a vertical plane that can be changed by dragging points and the the intersection between the sphere and a plane parallel to this vertical plane is got as o locus of vertical segments.
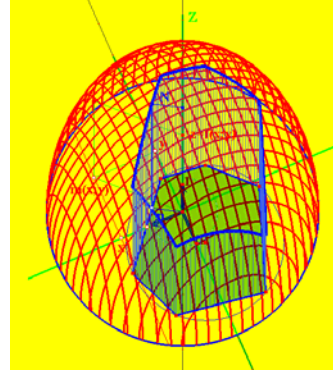
4.2. A cylinder

The intersection between the sphere and the cylinder is obtained as the locus of N when n moves on the circle which is the basis of the cylinder

To visualise the hole we have only asked the locus of segment [nN] whe n moves.

4.3. A prisma

This new intersection has been got after redefining point n as a point of a polygon which is the basis of the vertical prism
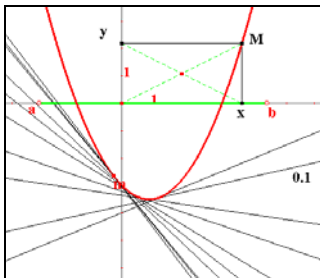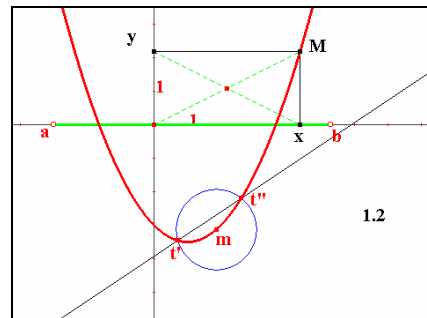
These examples show that students afford to solve them; they show that we can enrich the teaching of problems of intersections of surfaces closer from the genuine world. They show also that the use of a new instrument in Mathematics change Mathematics: new problems appears and justify the importance of the experimental way of practising mathematics obliging teacher to help their students to practice induction which was during a so long time a hidden stage of the Math activity; we begin only now to be interested with this stage even Polya, fifty years ago tried to tell us the importance of experiments to find

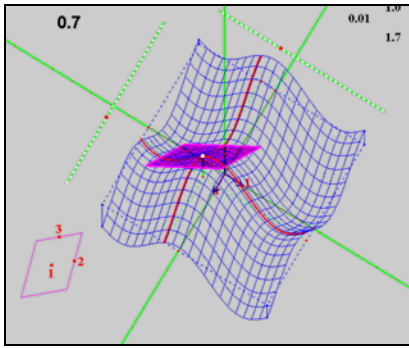## 5. Tangent planes to surfaces
5.1. Method

FIRSTLY, we have first created a macro giving the tangent line on a point belonging to a curve (which is a locus) called "**tangent line**".

Really it is not the tangent line but a secant line to the locus passing through point t' and t" depending from a positive number (here 1.2) that can be modified. It can be a good model of the tangent line when this number gets a small value like 0.1 or better 0.01

So we have done that here and validated the model in dragging m after asking Cabri to let the trace of this secant line on.
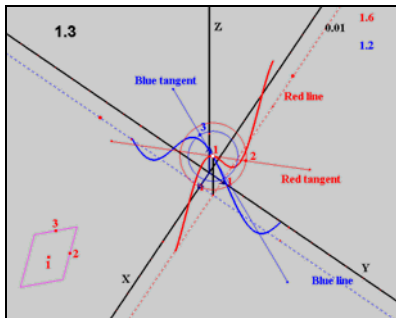
SECONDLY, we have created a macro constructing a parallelogram starting from three special points (have a look on the following figure)

We have used twice the first macro to the generic point 1 of the curve, one for the first red section and a other one for the second section. We have created a point on each tangent line 2 and 3.
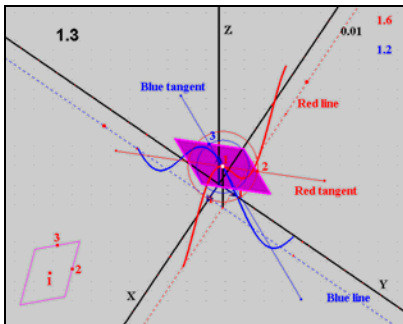
We have applied the second macro to these three points to get the purple parallelogram representing the tangent plane on the generic point to our surface.
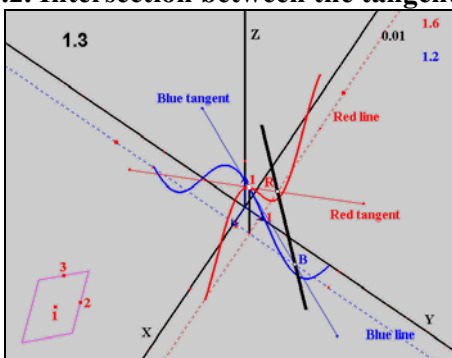
Let us see how we have done

We have drawn the tangent lines to the blue and the red loci in using the macro "**tangent line**" two times.
We used the tool compass to create points 2 and 3 on red and blue circles centered on 1 and having 1.6 and 1.2 as radius.
So we have got a system of axis of that tangent plane.

Here we have used the macro "**parallelogram from 3 points**" to get the purple parallelogram which is a representation of the tangent plane to the surface.

5.2. Intersection between the tangent plane and the horizontal plane

B and R are two particular points belonging to the tangent plane on the generic point of the surface.
B belongs to the tangent line of the blue section and also to the blue line which is horizontal.
R belongs to the tangent line of the red section and also to the red line which is horizontal.
The intersection of the tangent plane and the horizontal plane is drawn in thick and black ([BR])

We have decontextualised the notion of curve of function and tangent line in another geometric field, helping students to verify if they have really understood these notions when they are used out of the context they have been presented.

We have completed the parallelogram representing the tangent plane with two polygons to help our visualisation of the black intersection
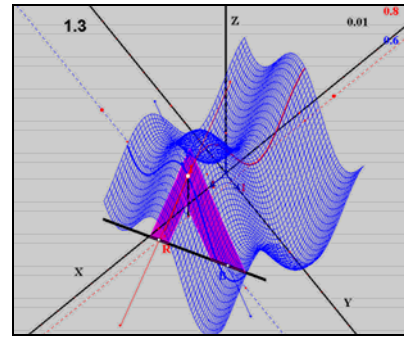
We ask for Cabri to give us again the loci representing the surface to get the representation on the left.

6. Conclusion

To approach the meaning of variable, I have proposed a special path ; my proposal was to use important contents of the curriculum of a lot of countries: I have linked the study of functions of two variables to the study of the ones of one variable in using the representations of surfaces in military perspective. I have used Cabri II Plus because it would have been impossible to do those things without it.

Softwares are the new instruments of Mathematics. This paper aimed to show that the math teacher must be both a specialist of Math and a specialist of the software he wants to use because a math instrument cannot be and will never be an artefact. The ways to use a software cannot be its user's booklet but the specific using allowing us to solve problems. I am working on a thesis about the experimental way of practising Math which will give the formal sequences of stages of this experimental way and show how the information tools enhance such a way to practice richer Mathematics.

References

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Géométrie avec cabri **CRDP de Grenoble** **P. Clarou C. Laborde, B. Capponi**
Première S Math book **Éditions Belin 1999** **Sous la direction de J.P. Bouvier avec J.J. Dahan**
Seconde Math book **Éditions Belin 2000** **Sous la direction de J.P. Bouvier avec J.J. Dahan**
Quelques difficultés d'apprentissage du concept de tangente **Repères IREM N°5** **M. Schneider-Gilot**
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Une autre manière d'enseigner la dérivation, la primitivation et l'intégration avec Cabri T3 2002 Calgary
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Images et perspectives **PNF Grenoble 05/2000** **J.J. Dahan M. Arragon**
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