

# INTERVAL ESTIMATION FOR QUANTILE ON ONE PARAMETER EXPONENTIAL DISTRIBUTION UNDER COMPLETE CENSORING WITH BOOTSTRAP PERCENTILE

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## Abstract

*In this article, two methods are proposed to give the interval estimation for the quantile on one parameter exponential distribution under complete censoring. Lawless (1982), Bain and Engelhardt (1992), Bury (1999) usually use traditional method to construct interval estimation. This interval needs an assumption that sample is exponentially distributed. We will consider an alternative method, known as the Bootstrap percentile to construct the confidence interval for the quantile of on one parameter exponential distribution. Bootstrap method has a better potential in constructing interval estimation compared to the traditional method. It gives shorter interval than the traditional method and this method does not need an assumption that the sample has to have an exponential distribution.*

## Introduction

The exponential distribution plays an important part in a lifetime distribution. Historically the exponential distribution was the first lifetime model in which statistical methods in survival analysis were extensively developed. What distinguishes survival analysis from other fields of statistics is censoring. Vaguely speaking, a censored observation contains only partial information about the random variable of interest. There are three types of censoring, that is the type I censoring, type II censoring and random censoring (Miller, 1981).

Some experiments are run over a fixed time period in such a way that an individual's lifetime will be known exactly only if it is less than some predetermined value. In such situations the data are said to be type I censoring or time censoring. A type II censored sample is one in which only the  $r$  smallest observations in a random sample of  $n$  items are observed. Censoring times are often effectively random. For example, in medical trial, patients may enter the study in a more or less random fashion, according to their time of diagnosis. If the study is terminated at some prearranged date, then censoring times, that is the lengths of time from an individual's entry into the study until the termination of the study, are random. If all  $n$  ordered observations are obtained, then this is called complete censoring (Lawless, 1982).

Lawless (1982) and Bury (1999) used traditional method to construct interval for quantile on one parameter exponential distribution under complete censoring. This interval needs an assumption that sample is random and have chi-square distributed. Bootstrap method is a computer-based methods for assigning measures of accuracy to statistical estimates, especially to calculate the confidence interval. The aim of using bootstrap method is to gain the best estimation from a minimal set of data (Efron and Tibshirani, 1993).

Fauzy and Ibrahim (2002a) have used bootstrap method in constructing the interval estimation for one parameter exponential distribution under complete censoring. Fauzy and Ibrahim (2002b) also used bootstrap method to construct for a survivor function of one parameter exponential distribution under complete censoring. In this paper the bootstrap percentile will be utilised to construct the interval estimation for quantile of a one parameter exponential distribution under complete censoring. The results will then compared to the traditional method to show its potentiality. The data in the text *Statistical Distributions In Engineering*, by Karl Bury (1999 page 200) will be used for illustration. An interval estimation for quantile of one parameter exponential distribution under complete censoring with traditional method will be constructed. Bootstrap's repeated results will be run to obtain the convergent condition. Subsequently, with this convergent condition, confidence interval for quantile of one parameter exponential distribution under complete censoring with bootstrap percentile method can be built.

### Probability Function

The actual survival time of an individual,  $t$ , can be regarded as the value of a variable  $T$ , which can take any non-negative value. The different values that  $T$  can take have a probability distribution, and we call  $T$  the random variable associated with the survival time. Now suppose that the random variable  $T$  has a probability distribution with underlying probability density function  $f(t)$ . The distribution function of  $T$  is then given by (Miller, 1981):

$$F(t) = P(T < t) = \int_0^t f(u) du \quad (1)$$

and represents the probability that the survival time is less than some value  $t$ .

### Complete Censoring

One parameter exponential distribution has probability density function (Lawless, 1982):

$$f(t; \mu, \theta) = \frac{1}{\theta} \exp\left(-\frac{t}{\theta}\right); \theta > 0 \quad (2)$$

Confidence intervals for  $\theta$  is:

$$\frac{2n\hat{\theta}}{\chi^2_{(1-\alpha/2, 2n)}} = \hat{\theta}_{\min} < \theta < \frac{2n\hat{\theta}}{\chi^2_{(\alpha/2, 2n)}} = \hat{\theta}_{\max} \quad (3)$$

The  $p$ th quantile of the distribution, given by  $t_p$ , where

$$t_p = [-\log(1-p)] \hat{\theta} \quad \text{with} \quad \hat{\theta} = \frac{\left(\sum_{i=1}^n T_{(i)}\right)}{n} \quad (4)$$

and the interval estimation for quantile ( $t_p$ ) is

$$[-(\log(1-p))\hat{\theta}_{\min}] < t_p < [-(\log(1-p))\hat{\theta}_{\max}] \quad (5)$$

### Bootstrap Percentile Method

In bootstrap method for setting confidence intervals and estimating significance levels it consists of approximating the distribution of a function of the observations and the underlying

distribution, such as a pivot, by what Efron calls the bootstrap distribution of this quantity. This distribution are obtained by replacing the unknown distribution by the empirical distribution of the data in the definition of the statistical function, and then resampling the data to obtain a Monte Carlo distribution for the resulting random variable (Bickel and Freedman, 1981).

Bootstrap method is a computer-based method for assigning measures of accuracy to statistical estimates, especially to calculate the confidence interval. Bootstrap itself comes from the phrase “*pull oneself up by one’s Bootstrap*”, which means stand up by your own feet and do things with minimal resources. The minimal resources are a minimum data, which is independent of any assumptions or data with no assumption at all about its population distribution. The aim of using bootstrap method is to get the best possible estimation from a minimal data.

The bootstrap’s percentile procedure for the interval estimation for quantile on one parameter exponential distribution under complete censoring are as follows:

1. give an equal opportunity  $1/n$  to every observation in the  $n$  data of complete censoring,
2. take  $n$  sample with replication,
3. do step 2 until B times in order to get an “*independent bootstrap replications*”,  $\hat{\beta}_n^{*1}, \hat{\beta}_n^{*2}, \dots, \hat{\beta}_n^{*B}$ , and search for convergent condition. Calculate the bootstrap  $t_p$  denoted by

$$t_p^{*i} = [-\ln(1-p)]\hat{\theta}_n^{*i}, \text{ with } \hat{\theta}_n^{*i} = \frac{\sum_{i=1}^n T_n^{*i}}{n} \quad (6)$$

4. define the confidence interval (of the bootstrap percentile) for quantile on one parameter exponential distribution under complete censoring as

$$\left[ t_p^{*i(\alpha/2)}, t_p^{*i(1-\alpha/2)} \right] \quad (7)$$

## Result And Discussion

As an illustration a secondary data from the text *Statistical Distributions in Engineering*, by Karl Bury (1999 page 200) was used. Sixteen units of a newly designed inverter were tested to failure under magnified vibratory loads, with the following results (in hours):

0.2, 0.5, 2.0, 2.3, 3.1, 4.5, 4.8, 7.0,  
10.1, 11.4, 12.0, 13.9, 18.0, 20.1, 21.5, 33.7

This is a complete censored data and distributed exponentially with one parameter. We will construct interval estimation for the 0.10 and 0.20 quantile ( $t_{0.10}$  and  $t_{0.20}$ ).

## Traditional Method

The interval estimation for a one parameter exponential distribution ( $\theta$ ) is

$$\frac{2n\hat{\theta}}{\chi_{(1-\alpha/2, 2n)}^2} = \hat{\theta}_{\min} < \theta < \frac{2n\hat{\theta}}{\chi_{(\alpha/2, 2n)}^2} = \hat{\theta}_{\max}$$

and the interval estimation for quantile ( $t_p$ ) is

$$[-(\log(1-p))\hat{\theta}_{\min}] < t_p < [-(\log(1-p))\hat{\theta}_{\max}]$$

The values of  $t_{0.10} = 1.087189$  and  $t_{0.20} = 2.302563$

Based on the above one parameter exponential distribution under complete censoring, we computed the interval estimations for the 0.10 and 0.20 quantile ( $t_{0.10}$  and  $t_{0.20}$ ). The results are in Table 1 and Table 2 respectively.

Table 1. The floor (F), ceiling (C) and interval width (IW) for the 0.10 quantile ( $t_{0.10}$ ) at the level of (L) 99 % and 95 %

L	F	C	IW
99 %	0.753125	1.733270	0.980145
95 %	0.940959	1.322609	0.381650

Table 2. The floor (F), ceiling (C) and interval width (IW) for the 0.20 quantile ( $t_{0.20}$ ) at the level of (L) 99 % and 95 %

L	F	C	IW
99 %	1.595047	3.670901	2.075854
95 %	1.992861	2.801160	0.808299

### Bootstrap Percentile

From bootstrap's repeated results we obtained the convergent condition at  $B = 6150$ . The plot of replication and bias in Figure 1 clearly showed this.

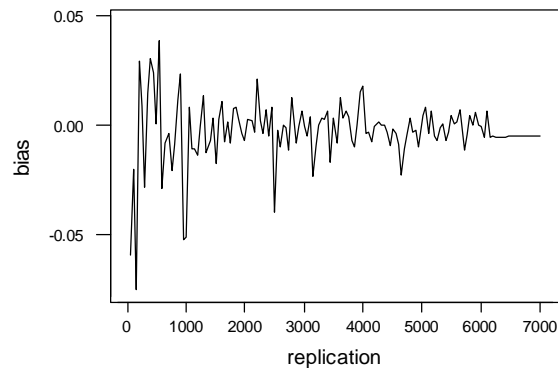


Figure 1. Plot between bias and replication

Estimation of the  $t_{0.10}$  and  $t_{0.20}$  at this replication are

$$t_{0.10} = 1.085881 \text{ and } t_{0.20} = 2.299793$$

After undergoing the bootstrap process, the floor (F), ceiling (C) and interval width (IW) for the 0.10 and 0.20 quantile ( $t_{0.10}$  and  $t_{0.20}$ ) at the level of (L) 99 % and 95 % are tabulated in Table 3 and Table 4.

Table 3. The floor (F), ceiling (C) and interval width (IW) for the 0.10 quantile ( $t_{0.10}$ ) at the level of (L) 99 % and 95 %

L	F	C	IW
99 %	0.716451	1.488217	0.771766
95 %	0.917953	1.243254	0.325301

Table 4. The floor (F), ceiling (C) and interval width (IW) for the 0.20 quantile ( $t_{0.20}$ ) at the level of (L) 99 % and 95 %

L	F	C	IW
99 %	1.517376	3.151903	1.634527
95 %	1.944138	2.633094	0.688956

Table 5 gives the widths of the confidence intervals for the 0.10 and 0.20 quantile on one parameter exponential distribution under complete censoring of the traditional method and the bootstrap percentile method.

Table 5. Widths of the intervals for the 0.10 and 0.20 quantile at level 99 % and 95 %

Method	$t_{0.10}$		$t_{0.20}$	
	99 %	95 %	99 %	95 %
Traditional	0.980145	0.381650	2.075854	0.808299
Bootstrap percentile	0.771766	0.325301	1.634527	0.688956
Difference interval	0.208379	0.056349	0.441327	0.119343

## Conclusion

Bootstrap percentile method gives a better results of the interval estimation for quantile on one parameter exponential distribution under complete censoring than the conventional method. It gives shorter interval and another advantage of this method is that it does not need any distribution assumption the estimator.

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