

# An Alternative to the Standard $R$ Chart

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## Abstract

The  $R$  chart is often used to monitor for changes in the process variability. However, the standard approach of plotting the range statistics,  $R_i$ ,  $i = 1, 2, \dots$  on the  $R$  chart is slow in detecting sustaining shifts of small magnitude in the process variability. The aim of this paper is to propose a more efficient alternative to the standard  $R$  control chart approach. The new alternative is based on the construction of a moving average control chart based on the  $R_i$  statistics. Briefly, the moving average of width  $w$  at time  $i$  is defined as simply the average of the  $w$  most recent subgroup ranges. Varying subgroup sizes will be considered. Comparisons in terms of the average run length (ARL) performances between the standard and the new approaches are made by means of simulation. All simulation programs are written in the SAS language. Note that ARL is defined as the average number of points that must be plotted on a control chart before an out-of-control signal is given. Since the new approach is more effective than the standard  $R$  chart approach, it may be an attractive alternative to the standard approach. This paper discusses how statistical software such as SAS is used in the evaluation of the performances of control charts.

## 1. Introduction

The aim of this paper is to propose a more superior alternative to the classical  $R$  chart for the detection of shifts in the process variability. The next section deals with the derivation of the  $R$  chart limits, followed by a discussion on the proposed alternative method in section 3. A comparison on the performances of the two different methods based on their ARL profiles will be given in section 4. Note that all the ARL profiles are computed using SAS version 6.12. In section 5, an example is given to illustrate how the proposed alternative method is put to work. In the conclusion section (see section 6), a brief discussion which summarizes the results of the paper is given.

## 2. Derivation of $R$ Chart Limits

The  $R$  chart control limits are based on the standard  $3\sigma$  approach, i.e.,  $\mu_R \pm 3\sigma_R$  where  $\mu_R$  and  $\sigma_R$  are the mean and standard deviation of the range statistic  $R$  (see Alwan, 2000; Duncan, 1974; Grant and Leavenworth, 1980; and Montgomery, 1996). The range statistic  $R$ , for each subgroup is the difference between the largest and the smallest observations in the subgroup. From Alwan (2000), the upper and lower control limits (denoted by  $UCL$  and  $LCL$  respectively) of the  $R$  chart are

$$\begin{aligned} UCL &= \bar{R} + 3d_3 \left( \frac{\bar{R}}{d_2} \right) \\ LCL &= \bar{R} - 3d_3 \left( \frac{\bar{R}}{d_2} \right) \end{aligned} \tag{1}$$

where  $\bar{R}$  is an estimate for  $\mu_R$  or, equivalently the expected value of  $R$ ,  $E(R)$ . Here,  $d_2$  and  $d_3$  are the control chart constants whose values depend on the sample size,  $n$  (see Appendix). By defining  $D_3 = 1 - \frac{3d_3}{d_2}$  and  $D_4 = 1 + \frac{3d_3}{d_2}$ , we obtain

$$\begin{aligned} UCL &= D_4 \bar{R} \\ LCL &= D_3 \bar{R} \end{aligned} \quad (2)$$

For the case where the standard deviation,  $\sigma$ , of the underlying distribution, assumed to be normal is known, and since  $\sigma = \frac{\bar{R}}{d_2}$ , then from (1) the limits for the  $R$  chart are

$$\begin{aligned} UCL &= \sigma d_2 + 3\sigma d_3 \\ LCL &= \sigma d_2 - 3\sigma d_3 \end{aligned} \quad (3)$$

### 3. The Design of a Moving Average $R$ Chart

Assume that all of the observations for each subgroup are independently and identically (i.i.d.) random observations with a  $N(\mu, \sigma^2)$  distribution. Also assume that all of the subgroups are independent of one another. If the ranges for a series of subgroups,  $R_1, R_2, \dots, R_i, \dots$ , are computed where  $R_i$  denotes the range of subgroup  $i$ , then the moving average statistic of width  $w$  at time  $i$  can be computed as

$$\begin{aligned} M_i &= \frac{R_i + R_{i-1} + \dots + R_{i-w+1}}{w} \\ &= \frac{\sum_{j=i-w+1}^i R_j}{w}, \quad i \geq w \end{aligned} \quad (4)$$

For periods  $i < w$ , there are not yet  $w$  ranges to calculate the moving average of width  $w$ . For these periods, the average of all the ranges up to period  $i$  defines the moving average at time  $i$ , i.e.,

$$M_i = \frac{\sum_{j=1}^i R_j}{i}, \quad i < w \quad (5)$$

For example, if  $w = 3$ , then

$$\begin{aligned} M_1 &= \frac{R_1}{1} = R_1 \\ M_2 &= \frac{R_1 + R_2}{2} \\ M_3 &= \frac{R_i + R_{i-1} + R_{i-2}}{3}, \quad i \geq 3 \end{aligned} \quad (6)$$

The mean of  $M_i$  for periods  $i \geq w$  is

$$E(M_i) = E\left[\frac{1}{w} \sum_{j=i-w+1}^i R_j\right]$$

$$\begin{aligned}
&= \frac{1}{w} E\left(\sum_{j=i-w+1}^i R_j\right) \\
&= \frac{1}{w} \sum_{j=i-w+1}^i E(R_j) \\
&= \frac{1}{w} (w\sigma d_2) \\
&= \sigma d_2
\end{aligned} \tag{7}$$

and its corresponding variance is

$$\begin{aligned}
Var(M_i) &= Var\left(\frac{1}{w} \sum_{j=i-w+1}^i R_j\right) \\
&= \frac{1}{w^2} Var\left(\sum_{j=i-w+1}^i R_j\right) \\
&= \frac{1}{w^2} (w\sigma^2 d_3^2) \\
&= \frac{\sigma^2 d_3^2}{w}
\end{aligned} \tag{8}$$

It can be shown easily that for periods  $i < w$ , the mean and the variance of the moving average are

$$E(M_i) = \sigma d_2 \tag{9}$$

and

$$Var(M_i) = \frac{\sigma^2 d_3^2}{i} \tag{10}$$

respectively.

The upper and lower control limits (denoted by  $UCL_M$  and  $LCL_M$  respectively) and the center line ( $CL_M$ ) for the moving average  $R$  chart for periods  $i \geq w$  are

$$\begin{aligned}
UCL_M &= \sigma d_2 + Ld_3 \sigma \left(\frac{1}{\sqrt{w}}\right) \\
CL_M &= \sigma d_2 \\
LCL_M &= \sigma d_2 - Ld_3 \sigma \left(\frac{1}{\sqrt{w}}\right)
\end{aligned} \tag{11}$$

and for periods  $i < w$ , the limits are

$$\begin{aligned}
UCL_M &= \sigma d_2 + Ld_3 \sigma \left(\frac{1}{\sqrt{i}}\right) \\
CL_M &= \sigma d_2 \\
LCL_M &= \sigma d_2 - Ld_3 \sigma \left(\frac{1}{\sqrt{i}}\right)
\end{aligned} \tag{12}$$

where  $L$  is the control chart constant whose value is determined based on the desired in-control ARL. For cases where the value  $\sigma$  is unknown, the limits in eqs. (11) and (12) are obtained by

substituting  $\sigma$  with its estimate  $\frac{\bar{R}}{d_2}$ , where  $\bar{R}$  is the average range estimated from a preliminary set of, say,  $m$  subgroup ranges and is given by

$$\bar{R} = \frac{\sum_{j=1}^m R_j}{m} \quad (13)$$

#### 4. Comparing the ARL Performances of $R$ and Moving Average $R$ Charts

ARL is an important feature in evaluating the performances of control charts. ARL represents the average number of points until a chart signals. Therefore, when there is no change in the process variability, a large ARL is desired so that less frequent false out-of-control signals are given. On the contrary, when the process variability increases to an undesirable level, a small ARL is desired to enable a quicker detection of the instability in the process.

The ARL profiles for the two types of charts are computed using SAS version 6.12. Here, we assumed that the in-control observations in each subgroup follow a  $N(\mu_0, \sigma_0^2)$  distribution whereas the out-of-control observations are  $N(\mu_0, \sigma_1^2)$  distributed, where  $\sigma_1 = \delta\sigma_0$  and  $\delta \in \{1.00, 1.05, \dots, 1.50, 1.75, 2.00, 2.50, \dots, 4.00, 5.00\}$ . The latter represents an increase in the process variability if  $\delta > 1$ . Note that if  $\delta = 1$ , there is no change in the process variability, thus, it is the in-control case. Subgroup sizes of  $n = 5$  and  $n = 10$  are considered. Their corresponding ARL profiles for the various schemes are displayed in Tables 1 and 2 respectively. Note that the values of  $L$  which are used in the computation of the moving average  $R$  chart limits in eqs. (11) and (12) are determined using simulation to achieve similar in-control ARL ( $ARL_0$ ) value to the  $R$  chart.

Table 1. ARL Profiles for  $R$  and Moving Average  $R$  Charts for  $n = 5$

$\delta$	$R$ Chart	Moving Average $R$ Chart		
		$w = 2 (L = 2.865)$	$w = 3 (L = 2.791)$	$w = 4 (L = 2.742)$
1.00	217.1	217.5	217.0	217.2
1.05	121.9	112.8	112.5	116.4
1.10	74.1	64.1	61.8	61.7
1.15	48.0	39.5	37.2	35.7
1.20	32.6	26.4	24.2	22.9
1.25	23.7	18.4	16.8	16.0
1.30	17.5	13.8	12.3	11.7
1.35	13.4	10.6	9.6	9.1
1.40	10.6	8.5	7.6	7.2
1.45	8.7	6.9	6.2	6.0
1.50	7.3	5.7	5.3	5.1
1.75	3.6	3.1	3.0	2.9
2.00	2.4	2.2	2.1	2.1
2.50	1.6	1.5	1.5	1.5
3.00	1.3	1.3	1.3	1.3
3.50	1.2	1.1	1.1	1.1
4.00	1.1	1.1	1.1	1.1
5.00	1.0	1.0	1.0	1.0

Table 2. ARL Profiles for  $R$  and Moving Average  $R$  Charts for  $n = 10$

$\delta$	$R$ Chart	Moving Average $R$ Chart		
		$w = 2$ ( $L = 2.885$ )	$w = 3$ ( $L = 2.818$ )	$w = 4$ ( $L = 2.770$ )
1.00	232.2	232.2	232.2	231.9
1.05	114.2	105.7	100.7	99.1
1.10	63.0	52.5	47.5	44.7
1.15	37.5	29.0	25.4	23.8
1.20	24.0	17.6	15.2	14.2
1.25	16.2	11.9	10.3	9.5
1.30	11.7	8.5	7.4	6.8
1.35	8.7	6.4	5.6	5.2
1.40	6.8	5.0	4.5	4.2
1.45	5.4	4.0	3.7	3.5
1.50	4.4	3.4	3.2	3.0
1.75	2.2	1.9	1.8	1.8
2.00	1.5	1.4	1.4	1.4
2.50	1.1	1.1	1.1	1.1
3.00	1.0	1.0	1.0	1.0

The results in both Tables 1 and 2 show that the ARL performances of the three ( $w = 2, 3$  and 4) schemes of the moving average  $R$  charts for  $\delta > 1$  are more superior to the classical  $R$  chart. For example, from Table 1, if  $\delta = 1.1$ , the three ARL values for  $w = 2, 3$  and 4 are all lower compared to that of the  $R$  chart. Therefore, the increase in the process variability ( $\delta > 1$ ) can be detected faster using the proposed moving average  $R$  chart for small to moderate shifts. However, as the magnitude of shift increases, say,  $\delta > 2$ , both control charts perform equally well. Since usually more concern is given to the detection of small shifts, the proposed method serves the purpose well.

## 5. Example

In this example, we will assume that the in-control observations follow a  $N(200, 4^2)$  distribution whereas the out-of-control observations follow a  $N(200, 5^2)$  distribution, i.e., the shift is due to an increase in the process variance. All of the observations are generated using a powerful statistical software, Minitab r13. The in-control observations consist of subgroups 1 to 20, each of size five while the out-of-control observations belong to subgroups 21 to 30. The number of observations in each of the ten out-of-control subgroups is also five.

Tables 3 and 4 show the in-control and the out-of-control observations for the 30 subgroups together with their corresponding ranges,  $R_i$ , and moving averages,  $M_i$ , for  $w = 2, 3$  and 4. The trial limits for the  $R$  and the moving average charts calculated based on the information in Table 3 are given in Table 5. Note that the values of the control chart constant,  $L$ , for the moving average  $R$  chart for  $w = 2, 3$  and 4 are chosen to achieve an in-control ARL of approximately 217. The values of  $L$  are also shown in Table 5. If a negative lower control limit is obtained from the computation using the formulas given in eqs. (11) and (12) then let  $LCL_M = 0$ . None of the  $R_i$  and  $M_i$  statistics in Table 3 fall outside their respective charts' limits shown in Table 5. Therefore, no revision is required. Thus, these trial limits can be used to monitor for future shifts in the process variance.

Table 3. In-Control Observations for the First 20 Subgroups Where Each Observation Follows a  $N(200,4^2)$  Distribution

Subgroup No., $i$	Observations					Range, $R_i$	Moving Average, $M_i$		
	$X_1$	$X_2$	$X_3$	$X_4$	$X_5$		$w = 2$	$w = 3$	$w = 4$
1	201.22	202.99	197.67	195.75	200.14	7.24	7.240	7.240	7.240
2	201.49	199.05	203.64	204.73	190.88	13.85	10.545	10.545	10.545
3	193.83	195.04	201.98	204.14	193.85	10.31	12.080	10.467	10.467
4	195.80	202.55	201.04	202.99	194.88	8.11	9.210	10.757	9.878
5	199.86	193.31	196.06	200.98	197.26	7.67	7.890	8.697	9.985
6	197.70	199.18	197.82	201.72	197.92	4.02	5.845	6.600	7.528
7	203.82	204.54	199.01	198.75	199.85	5.79	4.905	5.827	6.398
8	194.40	198.64	196.10	199.30	204.40	10.00	7.895	6.603	6.870
9	198.00	198.39	196.36	199.27	204.39	8.03	9.015	7.940	6.960
10	198.70	204.19	201.00	198.05	195.24	8.95	8.490	8.993	8.193
11	196.49	195.19	199.96	199.65	190.22	9.74	9.345	8.907	9.180
12	201.25	197.32	203.27	195.15	206.78	11.63	10.685	10.107	9.588
13	205.10	202.82	201.86	197.50	202.56	7.60	9.615	9.657	9.480
14	195.66	202.56	199.32	202.62	209.04	13.38	10.490	10.870	10.588
15	206.31	208.70	199.50	198.58	201.15	10.12	11.750	10.367	10.683
16	199.36	201.06	201.63	204.15	206.63	7.27	8.695	10.257	9.593
17	196.02	202.94	204.83	200.48	201.88	8.81	8.040	8.733	9.895
18	198.91	196.55	200.37	202.79	195.68	7.11	7.960	7.730	8.328
19	201.23	198.12	207.28	204.67	208.25	10.13	8.620	8.683	8.330
20	201.74	203.45	194.59	192.76	202.78	10.69	10.410	9.310	9.185

Table 4. Out-of-Control Observations for Subgroups 21 to 30 Where Each Observation Follows a  $N(200,5^2)$  Distribution

Subgroup No., $i$	Observations					Range, $R_i$	Moving Average, $M_i$		
	$X_1$	$X_2$	$X_3$	$X_4$	$X_5$		$w = 2$	$w = 3$	$w = 4$
21	205.92	208.56	200.72	201.64	200.13	8.43	9.560	9.750	9.090
22	199.67	191.19	202.10	206.87	199.74	15.68	12.055	11.600	11.233
23	196.55	213.29	197.52	196.07	196.08	17.22	16.450	13.777	13.005
24	196.62	201.42	208.58	208.90	204.37	12.28	14.750	15.060	13.403
25	201.74	195.08	189.98	204.74	203.84	14.76	13.52	14.753	14.985
26	200.01	196.57	196.59	199.37	200.64	4.07	9.415	10.370	12.083
27	189.63	205.62	201.94	195.68	189.45	16.17	10.120	11.667	11.820
28	209.06	198.49	201.75	206.61	197.59	11.47	13.820	10.570	11.618
29	205.99	200.81	195.94	208.58	204.49	12.64	12.055	13.427	11.088
30	197.69	199.93	201.95	202.86	202.16	5.17	8.905	9.760	11.363

Table 5. Trial Control Limits of the  $R$  and the Moving Average  $R$  Charts

$R$ Chart	Poisson Moving Average Chart		
	$w = 2$	$w = 3$	$w = 4$
<p>For <math>i \geq 1</math>,</p> $UCL = D_4 \bar{R}$ $= 2.115(9.0225)$ $= 19.08$ $CL = \bar{R} = 9.0225$ $LCL = D_3 \bar{R}$ $= 0$	<p>For <math>i = 1</math>,</p> $UCL_M = \bar{R} + 2.865 \bar{R} \frac{d_3}{d_2} \left( \frac{1}{\sqrt{1}} \right)$ $= 18.65$ $CL_M = \bar{R} = 9.0225$ $LCL_M = \bar{R} - 2.865 \bar{R} \frac{d_3}{d_2} \left( \frac{1}{\sqrt{1}} \right)$ $= 0$ <p>For <math>i \geq 2</math>,</p> $UCL_M = \bar{R} + 2.865 \bar{R} \frac{d_3}{d_2} \left( \frac{1}{2} \right)$ $= 15.81$ $CL_M = \bar{R} = 9.0225$ $LCL_M = \bar{R} - 2.865 \bar{R} \frac{d_3}{d_2} \left( \frac{1}{\sqrt{2}} \right)$ $= 2.23$	<p>For <math>i = 1</math>,</p> $UCL_M = \bar{R} + 2.791 \bar{R} \frac{d_3}{d_2} \left( \frac{1}{\sqrt{1}} \right)$ $= 18.38$ $CL_M = \bar{R} = 9.0225$ $LCL_M = \bar{R} - 2.791 \bar{R} \frac{d_3}{d_2} \left( \frac{1}{\sqrt{1}} \right)$ $= 0$ <p>For <math>i = 2</math>,</p> $UCL_M = \bar{R} + 2.791 \bar{R} \frac{d_3}{d_2} \left( \frac{1}{\sqrt{2}} \right)$ $= 15.64$ $CL_M = \bar{R} = 9.0225$ $LCL_M = \bar{R} - 2.791 \bar{R} \frac{d_3}{d_2} \left( \frac{1}{\sqrt{2}} \right)$ $= 2.41$ <p>For <math>i \geq 3</math>,</p> $UCL_M = \bar{R} + 2.791 \bar{R} \frac{d_3}{d_2} \left( \frac{1}{\sqrt{3}} \right)$ $= 14.42$ $CL_M = \bar{R} = 9.0225$ $LCL_M = \bar{R} - 2.791 \bar{R} \frac{d_3}{d_2} \left( \frac{1}{\sqrt{3}} \right)$ $= 3.62$	<p>For <math>i = 1</math>,</p> $UCL_M = \bar{R} + 2.742 \bar{R} \frac{d_3}{d_2} \left( \frac{1}{\sqrt{1}} \right)$ $= 18.21$ $CL_M = \bar{R} = 9.0225$ $LCL_M = \bar{R} - 2.742 \bar{R} \frac{d_3}{d_2} \left( \frac{1}{\sqrt{1}} \right)$ $= 0$ <p>For <math>i = 2</math>,</p> $UCL_M = \bar{R} + 2.742 \bar{R} \frac{d_3}{d_2} \left( \frac{1}{\sqrt{2}} \right)$ $= 15.52$ $CL_M = \bar{R} = 9.0225$ $LCL_M = \bar{R} - 2.742 \bar{R} \frac{d_3}{d_2} \left( \frac{1}{\sqrt{2}} \right)$ $= 2.52$ <p>For <math>i = 3</math>,</p> $UCL_M = \bar{R} + 2.742 \bar{R} \frac{d_3}{d_2} \left( \frac{1}{\sqrt{3}} \right)$ $= 14.33$ $CL_M = \bar{R} = 9.0225$ $LCL_M = \bar{R} - 2.742 \bar{R} \frac{d_3}{d_2} \left( \frac{1}{\sqrt{3}} \right)$ $= 3.72$ <p>For <math>i \geq 4</math>,</p> $UCL_M = \bar{R} + 2.742 \bar{R} \frac{d_3}{d_2} \left( \frac{1}{\sqrt{4}} \right)$ $= 13.62$ $CL_M = \bar{R} = 9.0225$ $LCL_M = \bar{R} - 2.742 \bar{R} \frac{d_3}{d_2} \left( \frac{1}{\sqrt{4}} \right)$ $= 4.43$

For the monitoring of future shifts, consider the data for the ten subgroups in Table 4. This is the out-of-control situation, thus a quick detection for shifts in the process variance is very important. Figures 1 – 4 show the  $R$  and the moving average control charts for  $w = 2, 3$  and  $4$  respectively incorporating all of the 30 subgroups. Although a permanent shift in the process variance occur from subgroup 21 onwards, the  $R$  chart fails to detect it. On the contrary, the moving average  $R$  chart for  $w = 2, 3$  and  $4$  give the first out-of-control signal at subgroups 23, 24 and 25 respectively. This example clearly shows the superiority of the moving average  $R$  chart over the classical  $R$  chart.

## 6. Conclusion

From the discussions in sections 4 and 5, it is evident that the moving average  $R$  chart is superior to the standard  $R$  chart. Due to this reason, it is recommended that the proposed moving average  $R$  chart be used as an attractive alternative by quality control practitioners.

## References

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## Appendix

Table 6. Factors for  $R$  Control Chart

Sample Size, $n$	$d_2$	$d_3$	$D_3$	$D_4$
2	1.128	0.853	0.000	3.267
3	1.693	0.888	0.000	2.574
4	2.059	0.880	0.000	2.282
5	2.326	0.864	0.000	2.115
6	2.534	0.848	0.000	2.004
7	2.704	0.833	0.076	1.924
8	2.847	0.820	0.136	1.864
9	2.970	0.808	0.184	1.816
10	3.078	0.797	0.223	1.777
11	3.173	0.787	0.256	1.744
12	3.258	0.778	0.283	1.717
13	3.336	0.770	0.307	1.693
14	3.407	0.763	0.328	1.672
15	3.472	0.756	0.347	1.653
16	3.532	0.750	0.363	1.637
17	3.588	0.744	0.378	1.622
18	3.640	0.739	0.391	1.609
19	3.689	0.734	0.404	1.597
20	3.735	0.729	0.415	1.585
21	3.778	0.724	0.425	1.575
22	3.819	0.720	0.435	1.566
23	3.858	0.716	0.443	1.557
24	3.895	0.712	0.452	1.548
25	3.931	0.708	0.459	1.541

(adapted from Alwan, 2000)