Simple Technique for Computing the Resistance of Cover-able Two-rooted Acyclic Digraph

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ABSTRACT

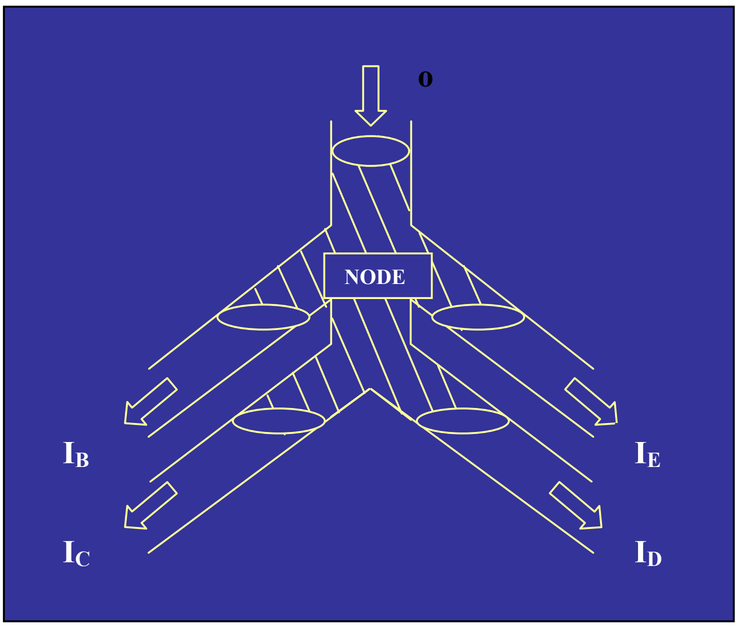
We consider acyclic two-rooted digraph in which the directed length of all paths between the two root points **o** and **r** has the same path length. Using the interface matrix technique we compute the resistance of such digraph and give a mathematical formulation of the simple formula which compute the resistance exactly in rational numbers.

KEYWORDS: Acyclic Digraph, Interface Matrix, Nodal graph, Non-nodal graph

1. Introduction and Notations

G.R Kirchhoff (1824-87) was the first who conducted important investigation concerning the distribution of current in electric circuits and known as Kichhoff's Current law. This fundamental results form the conservation of charge and it applies to a junction or node in circuit-- a point in the circuit where charge has several possible paths to travel(see for more details [9] and [7]). In Figure1, we see that I_A is the only current which flowing into the node. However there are four paths for current to leave the node namely I_B, I_C, I_D, I_E.When charge has entered into the node, it has no place to go except to leave.

Figure-1



In other words the total charge flowing into a node must be the same as the total charge flowing out of the node. Thus

$$I_B + I_C + I_D + I_E = I_A$$

or we can write

$$I_B + I_C + I_D + I_E - I_A = 0$$

Then, the sum of all the currents is zero and we can generalize it as below

$$\sum \mathbf{I} = 0$$

The circuit shown in Figure-1 is simple

We give here some definitions which we need to use in the rest of this paper. A digraph D consists of set of elements, called vertices and a list of ordered pairs of these elements, called arcs. The digraph with no loops or multiple arcs is called a simple digraph. A digraph D is connected if its underlying graph is a connected graph, and otherwise disconnected. A digraph D is acyclic if there is no cycle. A sequence of vertices from one vertex to another using the arcs is called directed path. The length of a directed path is the number of arcs used, or the number of vertices used minus one. A two-rooted acyclic digraph D is a simple connected digraph having two distinguish vertices **0** and **r** called roots, such that the length of all the directed connected paths between **0** and **r** must be same or constant. In figure-2, we give an example of two-rooted acyclic digraph with directed path length 5(constant)

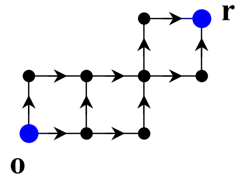


Figure-2: Example of an acyclic two-rooted digraph with directed path length 5 between roots **o** and **r.**

Much attention has been paid to complicated circuits and calculation of resistance (see [1],[2],[3] [4,],[5],[6]) for acyclic two-rooted digraphs. Because it is possible to calculate many other resistance related functions to be calculated for each directed edge. The purpose of this paper is to find a simple formula to calculate the resistance of directed graphs between two root points using the basic concept of Kirchhoff's law. The graphs here which will be considered are sub-graphs of the directed square lattice, the directed simple cubic lattice or the directed honeycomb lattice (for more details see [8]). We notice that for all these lattices the length of all directed paths from **o** to **r** is the same. We shall therefore assume in what follows that all directed paths between the roots of the graphs considered have the same length. For such a graph each vertex may be given a level number t which is the distance of the vertex from the first root. Let L(t) be the set of vertices in level s. For given vertex $k \in L(t)$, let $N^+(k)$ be the set of vertices adjacent to k in $L(t+1)$ and $N^-(k)$ be the set of vertices adjacent to k in $L(t-1)$. d_k^+ is the number of vertices in $N^+(k)$ adjacent to k in level t+1, and d_k^- is the number of vertices in $N^-(k)$.

We further need the following notation for use in the rest of this paper.

I_k = Total incoming current to the vertex k of L(t).

I'_ℓ = Total outgoing current from the vertex ℓ of L(t+1)

V_k = Negative potential of the vertex k of L(t).

V'_j = Negative potential of the vertex j of L(t+1) (to be calculated).

For j, k $\in$ L(t) , we define the conductance matrix σ_{kj} as

$$I_k = \sum_{j \in L(t)} \sigma_{kj} v_j \quad \text{.......... (1.1)}$$

and we denote the corresponding matrix for level t+1 by σ'.
Now we present the method in the section 2 to calculate the resistance.

2. Method Based on Interface Matrix

Using the Kerchhoff's current law defined above, we get the following equation

$$I_k = \sum_{j \in N^+(k)} V'_j - d^+_k V_k = \sum_{j \in L(t)} \sigma_{kj} V_j \quad \text{..........(2.1)}$$

Where σ is square matrix in which the diagonal elements are the values of conductance of the bond j of the given stage. On solving for V'_j, we get

$$\sum_{j \in N^+(k)} V'_j = \sum_{j \in L(t)} (\sigma_{kj} + d^+_k \ \delta_{kj}) V_j \quad \text{..........(2.2)}$$

To get V in terms of V' we must invert the matrix A where

$$A_{kj} = \sigma_{kj} + d^+_k \delta_{kj} \quad \text{..........(2.3)}$$

$$\sum_{j \in N^+(k)} (\sigma_{kj} + d^+_k \delta_{kj}) = \sum_{j \in N^+(k)} A_{kj} \quad \text{..........(2.4)}$$

Insert **A** to get V in terms of V' . Thus for k$\in$ L(t)

$$V_k = \sum_{m \in L(t)} (A^{-1}_{km}) \sum_{j \in N^+(m)} V'_j \quad \text{..........(2.5)}$$

But for $\ell \in$ L(t+1)

$$I'_\ell = d'_\ell v'_\ell - \sum_{k \in N^-(\ell)} v_k \quad \text{..........(2.6)}$$

and on combining this with equation 2.2, we get

$$I'_\ell = d^-_\ell v'_\ell - \sum_{k \in N^-(\ell)} \sum_{m \in L(t)} (A^{-1}_{km}) \sum_{j \in N^+(m)} v'j \quad \text{......................................}(2.7)$$

Here by definition of $\sigma'_{\ell n}$ equation 2.7 gives

$$\sigma'_{\ell n} = d'^-_\ell \ \delta_{\ell n} - \sum_{k \in N^-(\ell)} \sum_{m: n \in N^+(m)} (A^{-1}_{km}) \quad \text{................................}(2.8)$$

Since $n \in N^+(m) = m \in N^-$, so equation (2.7) may be written

$$\sigma'_{\ell n} = d'^-_\ell \delta_{\ell n} - \sum_{k \in N^-(\ell)} \sum_{m \in N^-(m)} (A^{-1}_{km}) \quad \text{..................................}(2.9)$$

Thus we have obtained a relation which determines in terms of σ. This may be used repeatedly starting with σ for stage 1, which is diagonal, and at the final stage t, σ is a (1 by 1) matrix giving the conductance of the whole graph. The above equation may be expressed in terms of an interface matrix, which gives the corresponding between level s+1 and s.

Let us define an interface matrix **F** as follows:

$$F_{mn} = \begin{cases} 1, & \text{if} \quad n \in N^+(m) \\ 0, & \text{otherwise} \end{cases} \quad \text{..}(2.10)$$

Now equation 2.7 may be written as follows

$$I'_\ell = d^-_\ell V'_\ell - \sum_{k \in N^-(\ell)} \sum_m (A^{-1}_{km}) \sum_n F_{mn} V'n \quad \text{...}(2.11)$$

and hence, we get

$$\sigma'_{\ell n} = d^-\delta_{\ell n} - \sum_k \sum_m F_{k\ell}(A^{-1}_{km})\ F_{mn} \quad \text{..}(2.12)$$

$$\sigma = \mathbf{d'} - \mathbf{B'}(\mathbf{A}^{-1})\mathbf{F} \quad \text{...}(2.13)$$

Where $\mathbf{A}^{-1} = (\sigma + \mathbf{d}^+)^{-1}$ ....(2.14)

Thus we have our equation to calculate conductance as follows

$$\sigma(s+1) = \mathbf{d}'^- - \mathbf{B'}[\sigma(s) + \mathbf{d}^+]^{-1}\mathbf{F} \ \text{ for } s = 1,\ldots,t-1 \quad \text{................}(2.15)$$

where σ **(s)** is the conductivity matrix for level s and σ **(1)** is unit matrix.

3. Example

To illustrate our method we calculate here the resistance of the two-rooted digraph which is shown in figure 3. In this digraph, there are four levels and the initial level, we put as

L(0)=1, and other are L(1)=3, L(2)=2, L(3)=2, and L(4)=1. Moreover, the directed path length between the two levels is same. It is readily seen that the path length between L(0) and L(4) is 4. The graph is directed in such a way that there is no circuit and it is acyclic. We suppose that the potential on the vertex in level 0 is zero and that each edge has unit resistance. By definition the first level conductance matrix is unit matrix.

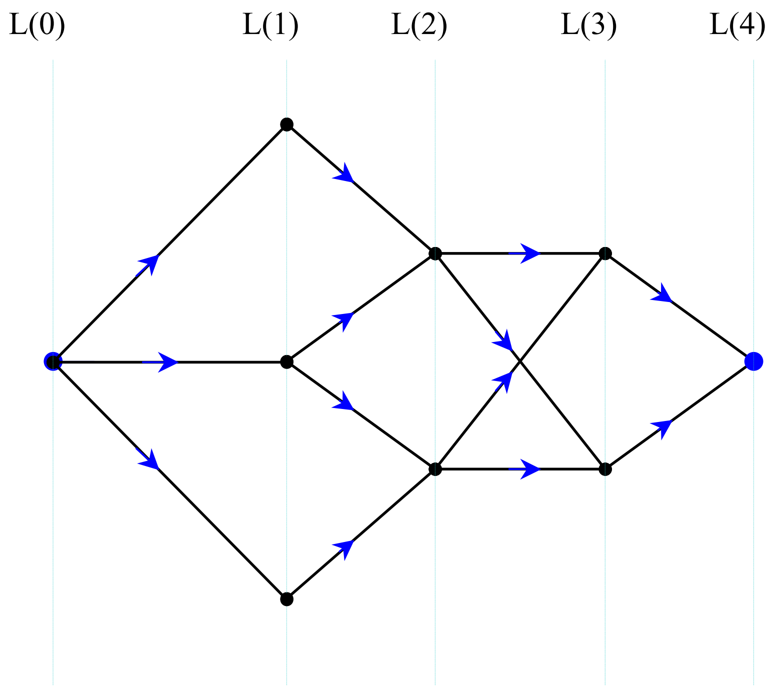


Figure- 3

$$\sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{for the level 1.}$$

Now consider the transition from level 1 to level 2 (i.e s = 1 above). In level 2 of the digraph there are two vertices. The first vertex in this level is adjacent to vertices 1 and 2 in level 1. Vertex 2 in this level is adjacent to vertices 2 and 3 in the level 1. Hence this level $\mathbf{d}'^{-1}$ is the 2 by 2 matrix.

$$\mathbf{d}'^{-} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

and

$$\sigma + \mathbf{d}^{+} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

As there are 3 vertices in level 1 and there are 2 vertices in level 2, therefore the matrix F is the following 3x2 matrix

$$\mathbf{F} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Now using, the equation 2.15, the conductance matrix for level 2 may be written as

$$\sigma' = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

On simplification we get the following matrix

$$\sigma' = \begin{bmatrix} \frac{7}{6} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{7}{6} \end{bmatrix}$$

Which will be used as the input matrix σ' for the next stage (s=2) of the calculation to which we now proceed.

In the level 3 there are again two vertices. Vertex 1 of this level is adjacent to vertices 1 and 2 in the previous level. Vertex 2 is adjacent to vertices 1 and vertices 2 in the previous level.

$$\mathbf{d}'^{-} = \mathbf{d}^{+} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

and the interface matrix **F** are as follows

$$\mathbf{B}' = \mathbf{F} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

The $\mathbf{A}^{-1}$ matrix for this stage is

$$(\sigma + \mathbf{d}^{+})^{-1} = \begin{bmatrix} \frac{7}{6}+2 & -\frac{1}{3} \\ -\frac{1}{3} & \frac{7}{6}+2 \end{bmatrix}$$

$$= \frac{2}{119} \cdot \begin{bmatrix} 19 & 2 \\ 2 & 19 \end{bmatrix}$$

using equation 2.15 we get the conductance matrix for level 3

$$\sigma' = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \frac{2}{119} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 19 & 2 \\ 2 & 19 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\sigma' = \begin{bmatrix} \frac{22}{17} & -\frac{12}{17} \\ -\frac{12}{17} & \frac{22}{17} \end{bmatrix}$$

In level 4 there is only one vertex. This vertex is adjacent to vertex 1 and vertex 2 in the previous level. Hence for this level $\mathbf{d}'^{-}$ is 1 by 1 matrix and may be written as

$$\mathbf{d}'^{-} = 2$$

and the interface matrix is as follows

$$\mathbf{F} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$[\sigma + \mathbf{d}^{+}]^{-1} = \left[\begin{bmatrix} 22/17 & -12/17 \\ -12/17 & 22/17 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]^{-1}$$

$$= \begin{bmatrix} 39/17 & -12/17 \\ -12/17 & 39/17 \end{bmatrix}^{-1}$$

$$\sigma' = 2 - [1 \quad 1] \begin{bmatrix} 39/17 & -12/17 \\ -12/17 & 39/17 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

On simplification, we get $\sigma' = \frac{20}{27}$ or the resistance between L[0] and L[4] is $R = \frac{27}{20}$.

4. Conclusion

We have described a simple method to compute the resistance of a acyclic two-rooted directed graph. The equation 2.15 is the powerful tool to calculate the conductivity of a graph. This method can be used to calculate the resistance of the non-nodal and nodal two-rooted directed sub-graphs of the directed square lattice and directed simple cubic lattice and it may also be used for the directed honeycomb lattice. Or, this method can be used for a two-rooted digraph which satisfies the criteria given in this paper. This method gives the resistance of the digraph under consideration in terms of rational number. It might be possible that other properties of a digraph can be found, because it is possible to calculate many other resistance related functions to be calculated for each directed edge.

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