On Annihilatingly Uniqueness of Directed Windmills

C.S. Gan and V.C. Koo
Faculty of Engineering and Technology
Multimedia University
Jalan Ayer Keroh Lama
75450 Malacca, Malaysia

Abstract

Let G be a digraph with n vertices and $A(G)$ be its adjacency matrix. A monic polynomial $f(x)$ of degree at most n is called an annihilating polynomial of G if $f(A(G)) = 0$. G is said to be annihilatingly unique if it possesses a unique annihilating polynomial. In this paper, the directed windmill $M_3(r)$ is defined and we study the annihilating uniqueness of $M_3(r)$.

1. Introduction

All graphs under consideration in this paper are directed, connected, finite, loopless and without multiple arcs. Undefined terms and notations can be found in [1] and [2].

By a *digraph* $G = (V,E)$, we mean a finite set V (the elements of which are called vertices) together with a set E of ordered pairs of elements of V (these ordered pairs are called arcs).

A *diwalk* in a digraph is an alternating sequence of vertices and arcs, $v_0, x_1, v_1, \ldots, x_k, v_k$ in which each arc x_i is (v_{i-1}, v_i). The length of such diwalk is k, the number of occurrences of arcs in it.

A *dicycle* C_k of order k is a digraph with vertex set $\{v_1, \ldots, v_k\}$ having arcs (v_i, v_{i+1}), $i = 1,2,\ldots, k-1$ and (v_k, v_1).

Let G be a digraph with n vertices. The *adjacency matrix* $A(G) = (a_{ij})$ of G is a square matrix of order n where the (i, j) entry, a_{ij}, is equal to the number of arcs starting at the vertex i and terminating at the vertex j. Let $A^k(G) = (a_{ij}^{(k)})$ where k is a positive integer and the (i, j) entry, $a_{ij}^{(k)}$ of $A^k(G)$ is the number of different diwalks at length k from the vertex i to vertex j.

The determinant of a square matrix A is denoted by $|A|$. The characteristic polynomial $|xI - A(G)|$ of the adjacency matrix $A(G)$ is called the *characteristic polynomial* of G and is denoted by $f(x)$. A monic polynomial $f(x)$ of degree at most n with $f(A(G)) = 0$ is called an *annihilating polynomial* of G. The existence of annihilating polynomial of G is guaranteed by its characteristic polynomial. G is said to be annihilatingly unique if it possesses a unique annihilating polynomial.

Annihilating uniqueness of digraphs are first studied by Lam and Lim (see [3] and [4]). Dicycles, dipaths, and diwheels are examples of annihilatingly unique digraphs.

The following results are well-known in linear algebra (see [5] and [6]).

Theorem 1 *Let A be an n × n matrix. If m(x) and f(x) are minimum polynomial and characteristic polynomial of A, respectively, then*

1. $f(A) = 0$.
2. *If f(x) is any polynomial with f(A) = 0, then m(x) divides f(x); in particular m(x) divides f(x).*
3. *Let* $\{x_1, x_2, \ldots, x_k\}$ *be the set of distinct eigenvalues of A, with* x_i *having algebraic multiplicity* c_i. *Then*

$$f(x) = (x - x_1)^{c_1}(x - x_2)^{c_2} \cdots (x - x_k)^{c_k}$$

and

$$m(x) = (x - x_1)^{q_1}(x - x_2)^{q_2} \cdots (x - x_k)^{q_k}$$

where the q_i *satisfies* $0 < q_i \le c_i$, *for* $i = 1, 2, \ldots, k$.

Furthermore, if k = n, then

$$m(x) = f(x) = (x - x_1)(x - x_2) \cdots (x - x_n)$$ ð

A matrix for which the minimum polynomial is equal to the characteristic polynomial is called *non-derogatory*; otherwise it is *derogatory*.

The following result is a consequence of Theorem 1.

Theorem 2 *The annihilating polynomial f(x) of any digraph G with adjacency matrix A(G) is unique if and only if A(G) is non-derogatory.* ð

A *linear directed graph* is a digraph in which each indegree and each outdegree is equal to 1, that is, it consists of dicycles.

As an example, G_1 and G_2 are two linear directed subgraphs of a digraph G with four vertices whereas G_3 is a linear directed subgraph of G with eight vertices (see Fig.1).

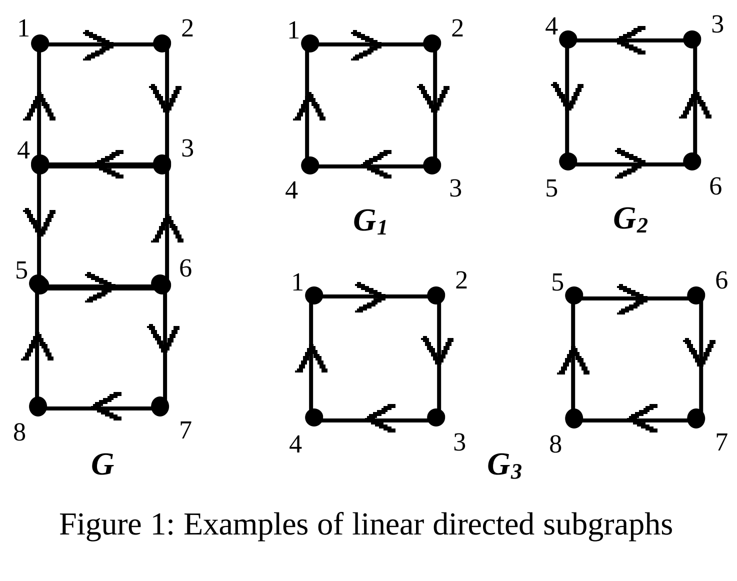

Figure 1: Examples of linear directed subgraphs

To find the characteristic polynomial of a digraph G, we quote the Coefficients Theorem for Digraphs from [1].

Theorem 3 ([1], *Theorem 1.2, pg.* 32) *Let* $x^n + a_1 x^{n-1} + \cdots + a_n$ *be the characteristic polynomial of a digraph G. Then for every i = 1,2, …,n ,*

$$a_i = \sum (-1)^{P(L)}$$

where the sum is taken over all linear directed subgraphs L (*i.e. directed subgraphs with only dicycles as components*) *of G with exactly i vertices; P(L) is the number of components in L.* ð

2. The Directed Windmills

The ***directed windmill*** *$M_3(r)$*, $r \geq 2$, (see Fig.2) is the digraph with $2r + 1$ vertices (labelled as 1,2, …, $2r+1$) together with arcs $(1,2k)$, $(2k+1,1)$ and $(2k,2k+1)$ for $1 \leq k \leq r$.

From the structure of $M_3(r)$, we see that it consists of r number of C_3 with a common vertex labelled 1.

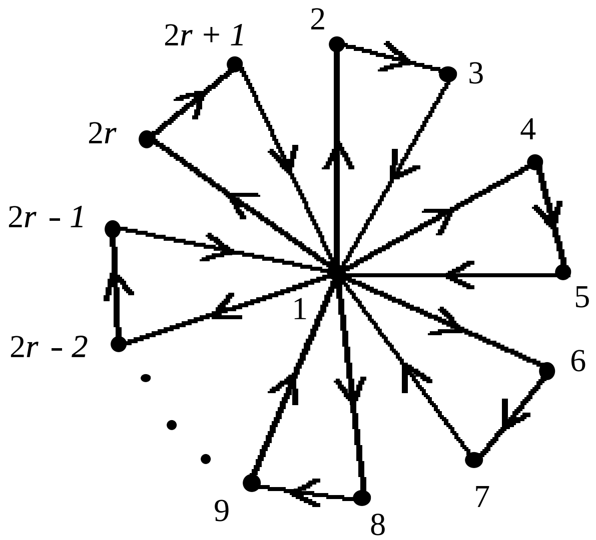

Figure 2: $M_3(r)$

Theorem 4 *$M_3(r)$ is annihilatingly unique if and only if* $r = 2$. ð

The proof of Theorem 4 is presented in next section.

3. Proof of Theorem 4

We shall prove Theorem 4 by using the result of Theorem 2, that is, we shall show that $A(M_3(r))$ is non-derogatory if and only if $r = 2$. It consists of a series of lemmas as follow:

Lemma 1 *If there exists at least one diwalk of length k from vertex i to vertex j in $M_3(r)$, then*

(i) $a_{ij}^{(k)} = 1$,
for $i \neq 1$ and when vertex 1 is excluded from the diwalk from vertex i to vertex j
or
the remaining diwalk of maximum length from vertex 1 to vertex j is less than 3;

(ii) $a_{ij}^{(k)} = r^t$ *when the remaining diwalk of maximum length from vertex 1 to vertex j is greater than or equal to 3q (q being a positive integer) and t is the number of C_3 contained in the remaining diwalk of maximum length from vertex 1 to vertex j.*

Proof. (i) is clear from the structure of $M_3(r)$. For (ii), suppose that the remaining diwalk of maximum length from vertex 1 to vertex j contains t number of C_3. Since there are r number of C_3 in $M_3(r)$, such a diwalk can start from vertex 1 of any C_3 of $M_3(r)$. Thus, we have $a_{ij}^{(k)} = r^t$. ð

Lemma 2 *The characteristic polynomial of $M_3(r)$ is given by* $f(x) = x^n - rx^{n-3}$, *where* $n=2r+1$ *is the number of vertices in $M_3(r)$.*

Proof. Let $f(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ be the characteristic polynomial of $M_3(r)$. Since $M_3(r)$ contains r number of C_3 with a common vertex labelled 1, we have $a_3 = -r$ and $a_i = 0$ for all $i \neq 3$ by Theorem 3 and the proof is complete. ð

Lemma 3 *$A(M_3(2))$ is non-derogatory.*

Proof. From Lemma 2, the characteristic polynomial of $M_3(2)$ is given by $f(x) = x^5 - 2x^2$ $= x^2(x^3 - 2)$. Let $g(x) = x^4 - 2x$ and A be the adjacency matrix of $M_3(2)$. From the structure of $M_3(2)$, we have $a_{43}^{(1)} = 0$ and $a_{43}^{(4)} = 1$. This implies that $g(A) = A^4 - 2A \neq 0$. Hence, we have $m(x) = f(x)$ and A is non-derogatory. ð

Lemma 4 *$A(M_3(r))$ with r ³ 3 is derogatory.*

Proof. Let $g(x) = x^{n-4}(x^3 - r) = x^{n-1} - rx^{n-4}$ where $n = 2r+1$. We shall show that $g(A) =$ $A^{n-1} - rA^{n-4} = 0$, where A is the adjacency matrix of $M_3(r)$ for r ³ 3.

If we assume that the number of C_3 in a diwalk of length $n-1$ from vertex i to vertex j is t, then the number of C_3 in a diwalk of length $n - 4$ ³ 3 is $t - 1$ for the corresponding pair of vertices. Note that if $n - 4 < 3$, that is, when $r < 3$, there exists no diwalk of length less than 3 from some vertices of i in one C_3 to some vertices j of another C_3 as illustrated in the case of $M_3(2)$.

From the structure of $M_3(r)$, if there exists a diwalk of length k from vertex i to vertex j, then there also exists a diwalk of length $k+3$ for the corresponding pair of vertices. By using Lemma 1, we have $a_{ij}^{(n-1)} - r\, a_{ij}^{(n-4)} = r^t - r(r^{t-1}) = 0$ for all i, j. Thus, we have $g(A) = 0$ and this implies that $m(x) \neq f(x)$. Hence A is derogatory. $\square$

4. Further Study

In general, we can extend the definition of directed windmills by defining $M_h(r)$, where $h \geq 4$ and $r \geq 2$ as the digraph with $(h-1)r+1$ vertices which consists of r number of C_h with a common vertex labelled 1. For example, $M_4(r)$ is illustrated in Fig.3:

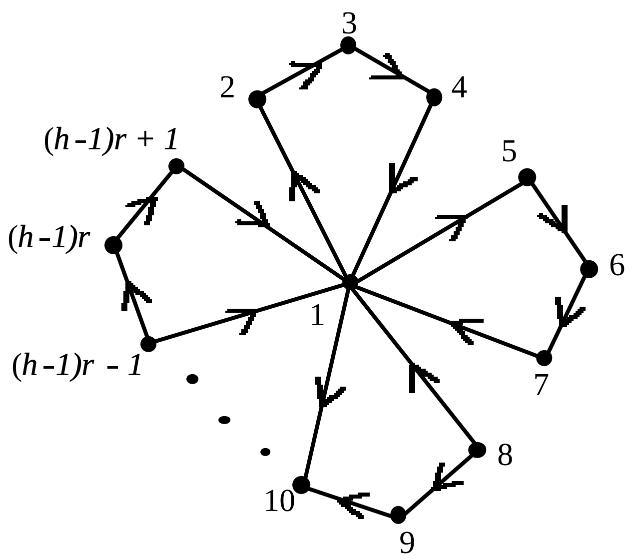

Figure 3: $M_4(r)$

By making necessary changes in Lemmas 1, 2, 3 and 4, we believe that $M_h(r)$, for $h \geq 4$, is also annihilatingly unique if and only if $r = 2$. However, a general proof for $h \geq 4$ and $r \geq 2$ is yet to be obtained.

References

[1] D.M. Cvetkovic, M. Dood and H. Sachs, *Spectra of graphs – Theory and applications,* Academic Press, New York, 1980.

[2] F. Harary, *Graph theory*, Addison-Wesley, Reading, MA, 1969.

[3] K.S. Lam, *On digraphs with unique annihilating polynomial,* Ph.D. thesis, University of Malaya, Kuala Lumpur, 1990.

[4] K.S. Lam and C.K. Lim, *The characteristic polynomial of ladder digraph and an annihilating uniqueness theorem*, Discrete Mathematics **151** (1996) 161-167.

[5] M. Marcus and H. Minc, *A survey of matrix theory and matrix inequalities*, Allyn and Bacon, Boston, 1964.
[6] E.D. Nering, *Linear algebra and matrix theory*, John Wiley and Sons Inc., New York, London, Sydney, Toronto, 1963.