Linear Transformation on the Points of the Unit Circle in the Plane and Its Visualization

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Abstract. In an elementary linear algebra class, students learn importance topics such as linear transformation and eigenvalues and eigenvectors. Especially students will gain more insight into linear transformations in the plane if they can see and interpret geometrically the effect on the plane of the linear transformations having 2 x 2 matrices as their representations. In particular the image of the unit circle under 2 x 2 real matrices is an ellipse, a line segment or a single point. Based on this fact we can get, via technology, an informative geometric picture and communicate some basic ideas of the geometry of linear transformation on the points of the unit circle, the eigenvalues and eigenvectors of a 2 x 2 real matrix and visualize the effect of circulation of a vector field on the unit circle.

Introduction

In teaching elementary linear algebra for students in their freshmen or sophomore year usually the objective is to present the fundamentals of linear algebra in the clearest possible way, and pedagogy is the main consideration.(see e.g H. Anton & C. Rorres (1994) and D.C. Lay (1998)). Many important concepts are given a geometric interpretation, because many students learn better when they can visualize an idea. At present, students have the opportunity to visualize geometric objects and analyze them in great details using a computer. The abilities to visualize and to calculate assume an equal status in the learning process (see e.g C.K. Cheung *et al*. (1996)). In this paper we shall view linear transformations in the plane as mapping points to points. And a useful way to visualizing the behavior of a linear transformation is to observe its effect on the points of simple figures in the plane. We shall see, and give some examples, the effect of some basic linear transformations in the plane on the points of the unit circle. Topic on eigenvectors and eigenvalues can also be easily grasp by students if we convert the algebraic meaning to a geometric picture. In the plane, eigenvectors and eigenvalues of a matrix A can also be found by testing many different vectors $\mathbf{x}$ until one turned up with the property that $A\mathbf{x}$ was parallel to $\mathbf{x}$. We shall see a visual realization of this relationship.

The image of the unit circle under 2 x 2 real matrices

Following P. Gathage and S. Shao (2001), let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2 x 2 nonsingular real matrix. The image of points of the unit circle under the matrix A will be given by $(x,y)^T = A\,(\cos t, \sin t)^T \quad (0 \le t \le 2\pi)$, and from this we have

$$(\cos t, \sin t)^T = A^{-1}(x,y)^T = \left(1/(ad-bc)\right)\left(dx - by, -cx + ay\right)^T .$$

Squaring the equations for cos t and sin t and then add, we obtain the following second-degree equation in x and y

$$\left(c^2+d^2\right)x^2-2(ac+bd)\,x\,y+\left(a^2+b^2\right)y^2=(ad-bc)^2.$$

We note $(ad-bc)=\det A\neq 0$ and $4\left(c^2+d^2\right)\left(a^2+b^2\right)-4(ac+bd)^2=4(ad-bc)^2>0$, which indicate that the graph of the above equation will be an ellipse with center at the origin. Let $B=\begin{pmatrix} c^2+d^2 & -(ac+bd) \\ -(ac+bd) & a^2+b^2 \end{pmatrix}$ be the symmetric coefficient matrix of the associated quadratic form of the above equation. We have $\det B=(ad-bc)^2$. Using the Principal Axis theorem, the above quadratic form can be diagonalized by a substitution $(x,y)^T=P\,(x',y')^T$ where P is a 2 x 2 orthogonal matrix. The jth column vector of P is a normalized eigenvector $\mathbf{p}_j$ of B corresponding to the eigenvalue λ_j of B. The diagonalized form appears as $\lambda_1 x'^2+\lambda_2 y'^2$ where λ_1 and λ_2 are the eigenvalues of the matrix B. These eigenvalues satisfies also $\det B=\lambda_1\lambda_2$. Hence the above substitution brings us to the ellipse $\dfrac{x'^2}{\lambda_2}+\dfrac{y'^2}{\lambda_1}=1$. In particular the area of this ellipse is given by

$\pi\sqrt{\lambda_1\lambda_2}=\pi\sqrt{\det B}=\pi\,|ad-bc|=\pi\,|\det A|=|\det A|$ x (area of the unit circle).

For the case where a 2 x 2 matrix A to be singular, its rank must be 0 or 1. If rank (A) = 0 then A is the zero matrix and we have $A\,(x,y)^T=(0,0)^T$, so all the points of the unit circle is mapped to a single point, the origin. If rank (A) = 1 then the column space of A is a one-dimensional subspace of R^2, which is a line through the origin. So the points of the unit circle is mapped to a line segment through the origin.

We note that the unit vectors in the plane can be represented by $\mathbf{u}=(\cos t,\sin t)^T$ $(0\leq t\leq 2\pi)$, and the unit circle in the plane with center at the origin can be regarded as a locus of the tips of the unit vectors that extend from the origin to the unit circle $x^2+y^2=1$ with center at (0,0) and radius 1. Given a 2 x 2 real matrix A, using computer we can plot simultaneously the vectors $\mathbf{u}(t)$ and $A\mathbf{u}(t)$ in the plane. For several values of t we can obtain several plot of $\mathbf{u}(t)$ and $A\mathbf{u}(t)$. If this is done for enough number of values of t distributed between 0 and 2π, it can be seen that the tips of $\mathbf{u}(t)$ generate a unit circle and the tips of $A\mathbf{u}(t)$ generate an ellipse in case the matrix A is non-singular or a line segment through the origin if rank (A) = 1 or a point (the origin) if rank (A) = 0.

A geometric aspect of linear transformations on the unit circle in the plane

We will visualize the effect on the points of the unit circle of some linear transformations having 2 x 2 real matrix as their representations. Let $A=\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, $B=\begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}$ and $C=\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Multiplication by the matrix A moves each point (x , y) parallel to the y-axis to the new position (x , -x+y). This is vertical shear with factor –1. In this case the matrix A is nonsingular and we obtain an ellipse as the image of the unit circle. The result of this transformation is shown in Figure 1. We note that the symmetric coefficient matrix of the associated quadratic form is $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and its eigenvalues are $\left(3 \pm \sqrt{5}\right)/2$. Hence the ellipse has $\sqrt{\left(3+\sqrt{5}\right)/2}$ as the length of its major axis and $\sqrt{\left(3-\sqrt{5}\right)/2}$ as the length of its minor axis.

Matrix B can be written as $B = \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Thus the matrix B represents the linear transformations that consists of vertical shear with factor 1, followed by a horizontal expansion with factor 2 and finally followed by a vertical expansion with factor 2. Since B is a non-singular matrix, again we obtain an ellipse as the image of the unit circle. The result of this transformation is shown in Figure 2. For this matrix B, its symmetric coefficient matrix of the

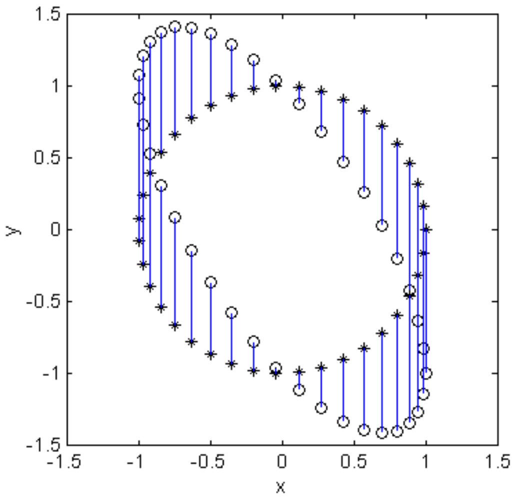

Figure 1

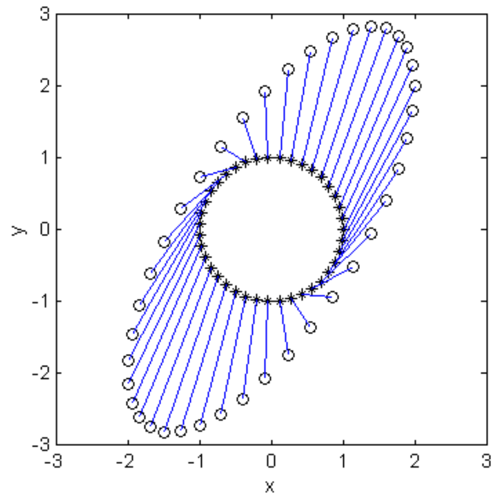

Figure 2

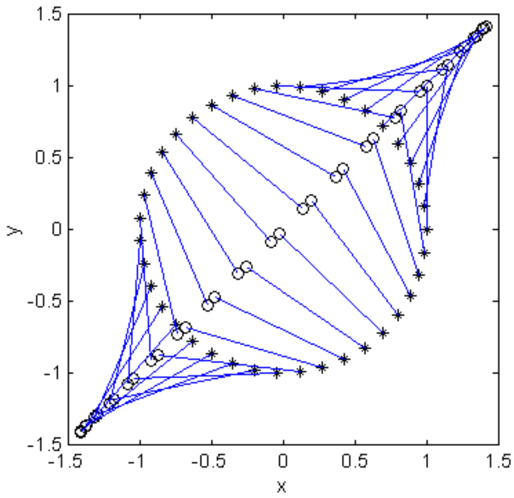

Figure 3

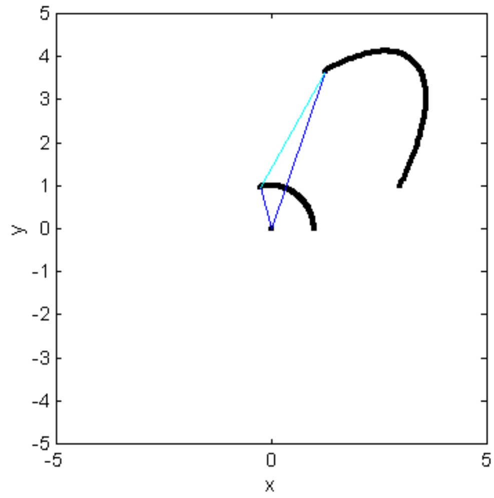

Figure 4

associated quadratic form is $\begin{pmatrix} 8 & -4 \\ -4 & 4 \end{pmatrix}$ and its eigenvalues are $-6 \pm 2\sqrt{5}$. Hence, the ellipse has $\sqrt{6+2\sqrt{5}}$ as the length of its major axis and $\sqrt{6-2\sqrt{5}}$ as the length of its minor axis.

Figure 3 shows the image of the unit circle under multiplication by the singular matrix $C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

In this case rank $(C) = 1$, hence we obtain a line segment through the origin as the image of the unit circle.

A visualization of eigenvectors and eigenvalues of 2 x 2 real matrices

We note that if A is a 2 x 2 matrix, than a nonzero vector $\mathbf{x}$ in R^2 is called an eigenvector of A if $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$; that is $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ and λ is an eigenvalue of A. Geometrically if λ is a real eigenvalue of A corresponding to $\mathbf{x}$ then multiplication by A dilates $\mathbf{x}$, contracts $\mathbf{x}$ or reverses the direction of $\mathbf{x}$, depending on the value of λ. In this case the vector $\mathbf{x}$ and $A\mathbf{x}$ will fall on the same line through the origin when plotted in the plane. Now look at again to our unit vectors $\mathbf{u}(t)$. Following C.R. Johnson and B.K. Kroschel (1998), if we take enough values for t which is distributed regularly over $[0, 2\pi]$ and combined with dynamic graphics, we can plot again simultaneously $\mathbf{u}(t)$ and $A\mathbf{u}(t)$ for each value of t. This time we see that the tips of the vector $\mathbf{u}(t)$ traverse out the unit circle as the tips of $A\mathbf{u}(t)$ traverse out an ellipse (or a line segment through the origin or a point, the origin, if A is a singular matrix) when t is allowed to vary from 0 to 2π. When the eigenvalues of A are real numbers, then there are values of t where the vectors $\mathbf{u}(t)$ and $A\mathbf{u}(t)$ fall on the same line through the origin. At this moment we identifies $\mathbf{u}(t)$ as an eigenvector of A and the length of the vector $A\mathbf{u}(t)$ as the absolute value of the eigenvalue of A corresponding to $\mathbf{u}(t)$. On the other hand if $\mathbf{u}(t)$ is an eigenvector of A than $-\mathbf{u}(t)$ is also an eigenvector of A. Hence in the case A is a nonsingular matrix, we identifies four positions where $\mathbf{u}(t)$ and $A\mathbf{u}(t)$ fall on the same line through the origin. We can also observe that when the two eigenvalues of A have the same sign the vectors $\mathbf{u}(t)$ and $A\mathbf{u}(t)$ traverse on the same counter clockwise direction, while if the two

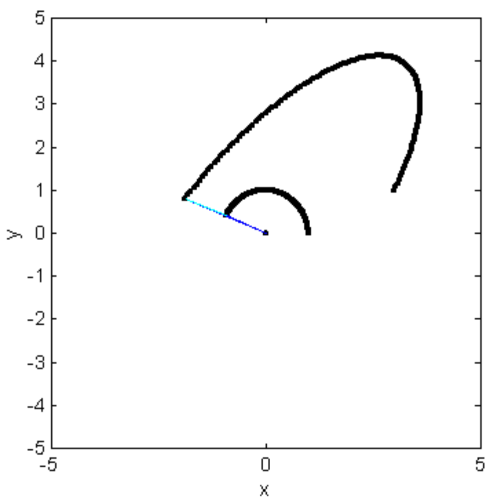

Figure 5

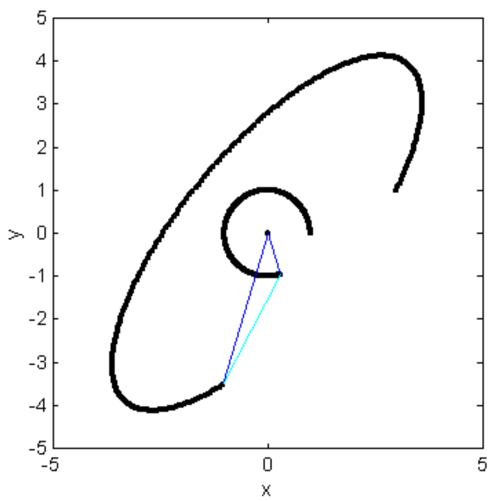

Figure 6

eigenvalues of A have opposite sign the two vectors traverse on the opposite direction. Figures 4, 5 and 6 show a process of mapping of the unit circle by the nonsingular matrix $A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$ which has 2 and 5 as its eigenvalues. In Figure 5 we see the eigenvector $\mathbf{u} = \left(-2/\sqrt{5}\,,1/\sqrt{5}\right)^T$ which correspond to the eigenvalue 2, fall on the same line and direction with the vector $A\mathbf{u} = \left(-4/\sqrt{5}\,,2/\sqrt{5}\right)^T$.

When we have a 2 x 2 real matrix with complex eigenvalues, we observe that the vectors $\mathbf{u}(t)$ and $A\mathbf{u}(t)$ never fall on the same line. In particular if $C = \begin{pmatrix} p & -q \\ q & p \end{pmatrix}$ is a real matrix with $pq \neq 0$ then its eigenvalues are the complex numbers $\lambda = p \pm iq$. If $r = \sqrt{p^2 + q^2}$ then C can be written as $C = r\begin{pmatrix} p/r & -q/r \\ q/r & p/r \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Hence the matrix C represents a linear transformation that consist of a rotation through the angle θ followed by a scaling with factor r. So for this matrix C the vectors $\mathbf{u}(t)$ and $C\mathbf{u}(t)$ never fall on the same line. More generally the real matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has complex eigenvalues if $(a-d)^2 + 4bc < 0$. If we write $p = (a+d)/2$ and $q = (1/2)\sqrt{-(a-d)^2 - 4bc}$, then the complex eigenvalues of A are $\lambda_1 = p - iq$ and $\lambda_2 = p + iq$ with the corresponding complex eigenvectors $\mathbf{v}_1 = (b, p-a)^T - i(0,q)^T$ and $\mathbf{v}_2 = (b, p-a)^T + i(0,q)^T$. Furthermore if $P = \begin{pmatrix} \mathrm{Re}\,\mathbf{v}_1 & \mathrm{Im}\,\mathbf{v}_1 \end{pmatrix}$ and $C = \begin{pmatrix} p & -q \\ q & p \end{pmatrix}$ then A can be written as $A = PCP^{-1}$. Hence we have a rotation "inside" the real matrix A having a complex eigenvalue (D.C. Lay, 1998).

A visualization of a circulation of a vector field on the unit circle

We consider now the mapping of the unit vectors $\mathbf{u}(t)$ by the matrix $I + A$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The resulting vectors which are $\mathbf{u} + A\mathbf{u}$ can be given the following interpretation. The units vectors $\mathbf{u}(t)$ can be associated with spokes of a wheel C and the vectors $A\mathbf{u}(t)$ at the end of the spokes as forces applied at the wheel's rim (P. Zizler and H. Fraser, 1997, J.F Dumais, 1982). With this interpretation we see that for the matrix $A = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$ such that $I + A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, the wheel in Figure 1 will turn clockwise in response to the forces $A\mathbf{u}(t)$. On the other hand for the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ such that $I + A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, the wheel in Figure 3 will not turn. The same is true for the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ such that $I + A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, the wheel in Figure 7 will not turn.

We will see that if A is a symmetric matrix than the forces $A\mathbf{u}$ can not produce a rotation on the wheel C. The circulation of $A\mathbf{u}$ around the unit circle C is given by the line integral around C of

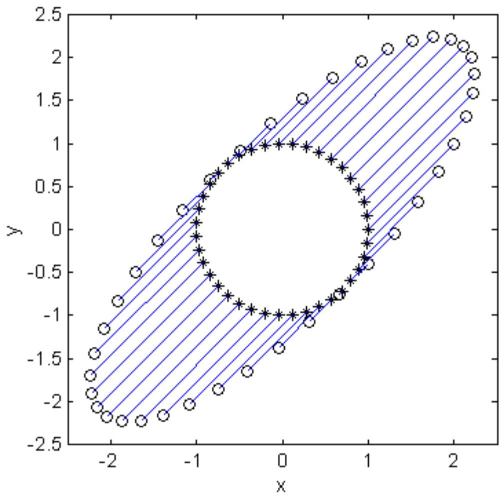

Figure 7

the tangential component of $A\mathbf{u}$, that is $\oint_C A\mathbf{u}\,.\mathbf{T}\,ds$; it measures the extent to which the corresponding forces $A\mathbf{u}$ produce a rotation around the circle C (see e.g. E. Kreyszig, 1988). With $\mathbf{u}(t) = (\cos t, \sin t)^T$, C: $\mathbf{r}(t) = (\cos t, \sin t)^T$ $(0 \le t \le 2\pi)$ and $\mathbf{T} = d\mathbf{r}/ds = (-\sin t, \cos t)^T\,(dt/ds)$ we obtain

$$\oint_C A\mathbf{u}.\mathbf{T}\,ds = \int_0^{2\pi} \mathbf{T}^T A\mathbf{u}\,dt = \int_0^{2\pi} (-\sin t, \cos t)(a\cos t + b\sin t,\, c\cos t + d\sin t)^T\,dt = \pi\ (c-b)$$

and its value will be zero if $b = c$, that is when A is a symmetric matrix. Hence the forces $A\mathbf{u}$ will not produce a rotation on the wheel C if A is a symmetric matrix.

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