Image Feature Analysis By Hahn Orthogonal Moments

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Abstract

Recently the Tchebichef and Krawtchouk moments have been introduced and studied by Mukundan, Ong and Lee ([1], [2]) and Yap, Raveendran and Ong ([3]) respectively. These moments have been shown to be useful as pattern features in the analysis of two-dimensional images. An advantage of these moments is that their implementation does not involve any numerical approximation, since the basis set is discrete in the domain of the discretized image coordinate space. This paper introduces a new set of orthogonal moment functions, the Hahn moments, based on the discrete Hahn orthogonal polynomials. The Hahn moments generalize the Tchebichef and Krawtchouk moments. The paper also details a number of theoretical properties of the Hahn moments useful in feature representation. In particular, limiting and Bayesian connections between the weighting functions of these moments are examined.

1. Introduction

In image analysis moment functions are used as shape descriptors in various applications like invariant pattern recognition, object classification and identification and robot vision. For an image intensity function $f(x, y)$, moment functions Φ_{pq} of order $(p+q)$ are defined as follows:

$$\Phi_{pq} = \int_x \int_y \Psi_{pq}(x,y) f(x, y)\, \mathrm{d}x\, \mathrm{d}y, \qquad p, q = 0,1,2,3....$$

where $\Psi_{pq}(x,y)$ is a continuous function of (x, y) known as the *moment weighting kernel* or the *basis set*. Hu [4] introduced the geometric moment functions given by

$$\Psi_{pq}(x,y) = x^p y^q$$

in order to derive shape descriptors invariant with respect to image plane transformations. Teague [5] proposed the Legendre and Zernike moments with the corresponding orthogonal functions as kernels. These orthogonal moments have better feature representation capabilities and are also less sensitive to image noise compared to geometric moments. An important feature of an orthogonal moment set is that information redundancy is a minimum.

However, the computation of orthogonal moments of images based upon continuous kernels possess two problems [6]: (a) discrete approximation of the continuous integrals (1), and (b) the normalization of image coordinate space to the domain of the orthogonal polynomials, which is either the range [-1, 1], or the interior of a unit circle. These difficulties motivated Mukundan, Ong and Lee ([1], [2]) and Yap, Raveendran and Ong ([3]) to consider discrete orthogonal polynomials as the basis functions for image moments. These authors introduced the Tchebichef moments and the Krawtchouk moments respectively.

The present paper introduces a new set of moment functions based on the Hahn orthogonal polynomials. The Hahn moments generalize the Tchebichef and Krawtchouk moments. In section 2 the Tchebichef and Krawtchouk moments are reviewed. The ensuing section introduces Hahn moments and a number of theoretical properties useful in feature representation. In particular, limiting and Bayesian connections between the weight functions of these moments and error analysis are examined.

2. Tchebichef and Krawtchouk Moments

2.1 Tchebichef Polynomials and Tchebichef Moments

The classical Tchebichef polynomials [3] are defined as

$$t_n(x) = (1-N)_n\, {}_3F_2(-n,-x,1+n;1,1-N;1) \qquad n, x, y = 0,1,2,..N-1. \tag{1}$$

where $(a)_k$ is the Pochhammer symbol given by

$$(a)_k = a\,(a+1)\,(a+2)\,\ldots(a+k-1), \tag{2}$$

and ${}_3F_2(.)$ is the generalized hypergeometric function,

$${}_3F_2(a_1,a_2,a_3;b_1,b_2;z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k (a_3)_k}{(b_1)_k (b_2)_k} \frac{z^k}{k!}. \tag{3}$$

The Tchebichef polynomials satisfy the orthogonality property

$$\sum_{x=0}^{N-1} t_m(x) t_n(x) = \rho(n,N)\delta_{mn}, \qquad 0 \le m, n \le N-1 \tag{4}$$

with

$$\rho(n,N) = \frac{N(N^2-1)(N^2-2^2)\ldots(N^2-n^2)}{2n+1} \qquad n = 0, 1, \ldots, N-1,$$

$$= (2n)!\binom{N+n}{2n+1},$$

and have the following recurrence relation:

$$(n+1)t_{n+1}(x)-(2n+1)(2x-N+1)t_n(x)+n(N^2-n^2)t_{n-1}(x)=0,$$
$$n=1, 2\ldots, N-1. \qquad (5)$$

We define the scaled Tchebichef polynomials as

$$\tilde{t}_n(x)=\frac{t_n(x)}{\beta(n,N)},$$

where $t_n(x)$ is the classical Tchebichef polynomial of order n, given by (1), and $\beta(n,N)$ is a suitable constant which is independent of x.

The Tchebichef moments are defined as

$$T_{pq}=\frac{1}{\tilde{\rho}(p,N)\tilde{\rho}(q,N)}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\tilde{t}_p(x)\tilde{t}_q(y)f(x,y) \qquad p, q = 0, 1, 2, ..N-1. \qquad (6)$$

The corresponding inverse moment transform is given by

$$f(x,y)=\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}T_{mn}\,\tilde{t}_m(x)\tilde{t}_n(y), \qquad x, y = 0, 1, 2, \ldots N-1. \qquad (7)$$

2.2 Krawtchouk polynomials and Krawtchouk Moments

The nth-order classical Krawtchouk polynomials are given by

$$k_n(x;p,N)=\frac{p^n}{n!}(-N)_n\ {}_2F_1\left(-n,-x;-N;\frac{1}{p}\right) \qquad (8)$$

corresponding to the weight function

$$w(x;p,N)=\binom{N}{x}p^x q^{N-x}, \qquad x=0,1,2,\ldots,N \qquad (9)$$

Another definition for Krawtchouk polynomials is given by Koekoek et al [7] as:

$$K_n(x;p,N)=\frac{n!}{p^n(-N)_n}k_n(x)=\ {}_2F_1\left(-n,-x;-N;\frac{1}{p}\right)$$

and the orthogonality condition is

$$\sum_{x=0}^{N}w(x;p,N)K_n(x;p,N)K_m(x;p,N)=\bar{\rho}(n;p,N)\delta_{nm}, \qquad (10)$$

where n, m=0, 1, 2, …, N

$$\bar{\rho}(n;p,N)=(-1)^n\left(\frac{1-p}{p}\right)^n\frac{n!}{(-N)_n} \qquad (11)$$

The weighted Krawtchouk polynomials $\{\overline{K_n}(x;p,N)\}$ is defined by setting

$$\overline{K_n}(x;p,N)=K_n(x;p,N)\sqrt{\frac{j(x;p;N)}{\phi(n;p,N)}} \qquad (12)$$

such that the orthogonality condition becomes

$$\sum_{x=0}^{N} \overline{K_n}(x;p,N)\overline{K_m}(x;p,N) = \delta_{nm}$$

For an image intensity function $f(x,y)$, the definition of Krawtchouk moments of order $(n+m)$ in terms of weighted Krawtchouk polynomials [3] (see also [1], p.1359) are given as

$$K_{nm} = \sum_{x=0}^{N-1}\sum_{y=0}^{N-1} \overline{K_n}(x;p_1,N-1)\overline{K_m}(y;p_2,N-1)f(x,y) \tag{13}$$

3. Hahn Polynomials and Moments

3.1 Hahn Polynomials

The *n*th-order classical Hahn polynomials [8] are given by

$$p_n(x;\beta,\gamma,\delta) = \frac{(\beta)_n(\gamma)_n}{n!}\ {}_3F_2(-n,-x;\beta+\gamma-\delta+n;\beta,\gamma;1)$$

corresponding to the weight function

$$w(x;\beta,\gamma,\delta) = \frac{(\beta)_x(\gamma)_x}{(\delta)_x\, x!}, \qquad x = 0,1,2,\ldots,N-1$$

However, in order to be in line with the development of the Krawtchouk moments, the following simpler definition [7] is adopted:

$$Q_n(x;\alpha,\beta,N) = {}_3F_2(-n,-x,n+\alpha+\beta+1;\alpha+1,-N;1),\ \alpha>-1,\beta>-1 \tag{14}$$

with weight function

$$w(x;\alpha,\beta,N) = \binom{\alpha+x}{x}\binom{\beta+N-x}{N-x}, \qquad x = 0,1,2,\ldots,N$$

The orthogonality condition is given by

$$\sum_{x=0}^{N} w(x;\alpha,\beta,N)Q_n(x)Q_m(x) = \rho(n;\alpha,\beta,N)\delta_{nm}\ , \tag{15}$$

where

$$\rho(n;\alpha,\beta,N) = \frac{(-1)^n(n+\alpha+\beta+1)_{N+1}(\beta+1)_n\, n!}{(2n+\alpha+\beta+1)(\alpha+1)_n(-N)_n\, N!}$$

Three-term recurrence relation and other properties are found in [7] and [8].

3.2 Hahn Moments

Following the definition of the Krawtchouk moments of order $(n+m)$ in terms of weighted Krawtchouk polynomials (13), we define the Hahn moments of order $(n+m)$ as

$$Q_{mn} = \sum_{x=0}^{N-1}\sum_{y=0}^{N-1} \overline{Q_m}(x;\alpha_1,\beta_1,N-1)\overline{Q_n}(y;\alpha_2,\beta_2,N-1)f(x,y) \tag{16}$$

where

$$\overline{Q_m}(x;\alpha,\beta,N-1) = \sqrt{w(x;\alpha,\beta,N-1)/\rho(m;\alpha,\beta,N-1)}\,Q_m(x;\alpha,\beta,N-1)$$

is the weighted Hahn polynomial.

4. Definition of Orthogonal Moments with respect to Weighted Image Intensity Function

It is of interest to note that in definition (16) (respectively (13))of the Hahn (Krawtchouk) moments the square root of the negative hypergeometric (binomial) weight function is used as a weight for the Hahn (Krawtchouk) polynomials. In contrast, for the Legendre and Zernike moments, the weight is unity. Yap et al ([3]) have demonstrated the role of the binomial weight in feature extraction by allowing the image to be focused at different sections according to varying values of the binomial parameter p. For instance, for the case of $p=1/2$, the binomial weight (9) is symmetric and the image is focused at the centre.

Note that if $m=n=0$ in (16), we have

$$Q_{00} = \sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\sqrt{w(x;\alpha_1,\beta_1,N-1)}\sqrt{w(y;\alpha_1,\beta_1,N-1)}f(x,y) \tag{17}$$

and this may be interpreted as the total weighted image intensity function. Alternatively, consider

$$Q_{00} = \sum_{x=0}^{N-1}\sum_{y=0}^{N-1} w(x;\alpha_1,\beta_1,N-1)w(y;\alpha_2,\beta_2,N-1)f(x,y) \tag{18}$$

which is easier to handle. An implication of (18) is that the orthogonal moments can be defined with respect to the *weighted* image intensity instead of the original image intensity. Therefore,

$$Q_{mn} = \sum_{x=0}^{N-1}\sum_{y=0}^{N-1} Q_m(x;\alpha_1,\beta_1,N-1)Q_n(y;\alpha_2,\beta_2,N-1)f^*(x,y) \tag{19}$$

where

$$f^*(x,y) = w(x;\alpha,\beta,N-1)w(y;\alpha,\beta,N-1)f(x,y)$$

The inverse moment transform is

$$f(x,y) = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} Q_{mn}Q_m(x)Q_n(y). \tag{20}$$

5. Connection between the Normalized Weight Functions and Limiting Cases

5.1 Connection between the Normalized Weight Functions

In this section we shall examine the connection between the weight functions for the three classes of orthogonal moments according to the variation of the binomial parameter p. This will be done with respect to the normalized weight functions.

Definition: The *normalized* form of the discrete weight function $w(x)$ is given by $w^*(x) = w(x)/\sum_x w(x)$.

Since $w^*(x) \ge 0$ and $\sum_x w^*(x) = 1$, the normalized weight function is a probability mass function. Systems of orthogonal polynomials are constructed or defined with respect to normalized or non normalized weight functions. The normalized weight functions for the three systems of orthogonal polynomials are tabulated below.

Polynomial	Weight $w(x)$	Normalized Weight $w^*(x)$
Tchebichef	1	$1/N$ (discrete uniform)
Krawtchouk	$\binom{N}{x} p^x q^{N-x}$	$\binom{N}{x} p^x q^{N-x}$ (binomial)
Hahn	$\binom{\alpha+x}{x}\binom{\beta+N-x}{N-x}$	$\binom{\alpha+x}{x}\binom{\beta+N-x}{N-x} \Big/ \binom{\alpha+\beta+N+1}{N}$ (negative inverse hypergeometric)

Table 1. Normalized Weight Functions for Tchebichef, Krawtchouk and Hahn polynomials

Note that the normalized weight functions are well-known probability mass functions. The Hahn polynomials, with respect to the normalized weight $w^*(x)$, have been considered by Weber and Erdelyi [9]. It is straightforward to recast formulae for the orthogonal polynomials in terms of the normalized weight functions. For instance, the orthogonality condition (4) of the Tchebichef polynomials may be written as

$$\sum_{x=0}^{N-1} w^*(N) t_m(x) t_n(x) = N^{-1} \rho(n,N) \delta_{mn} , \qquad 0 \le m, n \le N-1$$

where $w^*(N) = 1/N$ is the discrete uniform probability mass function (pmf).

Consider the weighted image intensity function (of the form (19)) with respect to the binomial weight with two parameters N and p. Since N represents the number of pixels, p is the only unknown parameter in the binomial weight. Information about p may be considered in a Bayesian context, by regarding p as a random variable. A natural prior distribution for the binomial parameter p is the beta distribution having probability density function

$$f(p) = p^{\gamma-1}(1-p)^{\delta-1} / B(\gamma,\delta) \qquad 0 < p < 1 ,$$

where $B(\gamma,\delta) = \Gamma(\gamma)\Gamma(\delta)/\Gamma(\gamma+\delta)$. The marginal (or prior predictive) distribution is

$$w(x) = \int_0^1 w(x; N \mid p) f(p)\, dp = \int_0^1 \binom{N}{x} p^x q^{N-x} f(p)\, dp$$

$$= \frac{\binom{\gamma+x-1}{x}\binom{\delta+N-x-1}{N-x}}{\binom{\gamma+\delta+N-1}{N}} \qquad (21)$$

on employing the formula

$$\binom{-a}{n} = (-1)^n \frac{(a)_n}{n!}.$$

It is easy to see that (21) is the normalized weight in Table 1 if we let $\gamma - 1 = \alpha,\ \delta - 1 = \beta$.

If $\gamma = 1$ and $\delta = 1$, $w(x) = 1/N$ is the discrete uniform distribution. In this case, $f(p) = 1,\ 0 < p < 1$, the continuous uniform distribution over the interval $(0,1)$. Thus a continuous uniform prior for the binomial parameter p leads to the discrete uniform prior predictive, and the corresponding Tchebichef polynomials and orthogonal moments. A beta prior gives the negative inverse hypergeometric distribution resulting in the Hahn polynomials and moments.

5.2 Limiting Cases and Similarity between the Orthogonal Moments

Mukundan et al [1] commented that the remarkable similarity between the values of Tchebichef moments and Legendre moments is a consequence of the limiting relation

$$\lim_{N\to\infty} N^{-n}\, t_n(Nx) = P_n(2x-1),\quad 0 < x < 1, \tag{22}$$

where $P_n(x)$ is the Legendre polynomial. We now discuss some limiting results involving the Hahn, Krawtchouk and Tchebichef moments.

A well-known limiting form of the Hahn polynomial is as follows:

If $\alpha = pr,\ \beta = qr$ and $0 < p = 1 - q < 1$, then $\lim_{r\to\infty} Q_n(x; pr, qr, N) = k_n(x; p, N)$.

Furthermore, under these conditions the Hahn weight function

$$w^*(x; \alpha, \beta, N) \to w^*(x; p, N)$$

It may be inferred that for large values of α and β the Hahn moments should be very similar to the Krawtchouk moments. On the other hand, the discussion in the preceding section shows that as α and β both tend to zero (that is, they are very small), the Hahn moments approach the Tchebichef moments. A consequence of these results is that the more general class of Hahn orthogonal moments may be used and the magnitude of the values of α and β will determine the appropriate class of orthogonal moments, Tchebichef, Krawtchouk or Hahn.

A generalization of (22) is the following limiting result of Weber and Erdelyi [9]:

$$\lim_{N\to\infty} Q_n((N-1)x; \alpha, \beta, N) = P_n^{(\alpha,\beta)}(1-2x) / P_n^{(\alpha,\beta)}(1) \tag{23}$$

where

$$P_n^{(\alpha,\beta)}(t) = \binom{\alpha+\beta}{n}\ {}_2F_1(-n, \alpha+\beta+n+1; n+1; (1-t)/2) \tag{24}$$

is the Jacobi polynomial. The Legendre polynomial is a special case if $\alpha = 0,\ \beta = 0$.

6. Error Analysis

In image reconstruction, the orthogonal moments are used up to an order *t*:

$$\hat{f}(x,y)=\sum_{m=0}^{t}\sum_{n=0}^{t}Q_{mn}Q_m(x)Q_n(y)\,.$$

The error in the reconstruction due to the truncation may be measured by

$$E=\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}w(x)\,w(y)\left(f(x,y)-\hat{f}(x,y)\right). \tag{25}$$

The following result gives a bound for the truncation error (25) which is useful to gauge the accuracy of the image reconstruction.

Result.
For $\max\{\alpha,\beta\}>-1/2$, the truncation error has the following bound

$$|E|<\sum_{m=t}^{N-1}\sum_{n=t}^{N-1}|Q_{mn}|\max\left\{\left|P_m^{(\alpha_1,\beta_1)}(-1)\right|,\left|P_m^{(\alpha_1,\beta_1)}(1)\right|\right\}\max\left\{\left|P_n^{(\alpha_2,\beta_2)}(-1)\right|,\left|P_n^{(\alpha_2,\beta_2)}(1)\right|\right\} \tag{26}$$

where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial defined by (24).

Proof.
Consider equation (25) after substituting for $f(x,y)$ and $\hat{f}(x,y)$:

$$\begin{aligned}E&=\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\{\sum_{m=t}^{N-1}\sum_{n=t}^{N-1}Q_{mn}w(x)Q_m(x)w(y)Q_n(y)\}\\&=\sum_{m=t}^{N-1}\sum_{n=t}^{N-1}Q_{mn}\{\sum_{x=0}^{N-1}w(x)Q_m(x)\}\{\sum_{y=0}^{N-1}w(y)Q_n(y)\}\end{aligned}$$

It is straightforward to prove the following result:

$$\sum_{x=0}^{N-1}\binom{N-1}{x}\rho^x(1-\rho)^{N-1-x}Q_n(x;\alpha,\beta,N-1)=P_n^{(\alpha,\beta)}(1-2\rho),\quad 0<\rho<1$$

If (21) is used in this result we get

$$\begin{aligned}&\sum_{x=0}^{N-1}\int_0^1\rho^{\alpha-1}(1-\rho)^{\beta-1}/B(\alpha,\beta)\binom{N-1}{x}\rho^x(1-\rho)^{N-1-x}Q_n(x;\alpha,\beta,N-1)d\rho\\&\qquad=\int_0^1P_n^{(\alpha,\beta)}(1-2\rho)\rho^{\alpha-1}(1-\rho)^{\beta-1}/B(\alpha,\beta)d\rho\,.\end{aligned}$$

Application of the well known inequality for the Jacobi polynomials

$$\left|P_n^{(\alpha,\beta)}(x)\right|<\max\left\{\left|P_n^{(\alpha,\beta)}(-1)\right|,\left|P_n^{(\alpha,\beta)}(1)\right|\right\},\qquad \max\{\alpha,\beta\}>-1/2,x\in[-1,1]$$

and the normalization for the beta distribution, namely,

$$\int_0^1\rho^{\alpha-1}(1-\rho)^{\beta-1}/B(\alpha,\beta)d\rho=1$$

lead to the result (26). □

The sharpness of the bound for the truncation error and related measures of the performance of the image reconstruction will be considered elsewhere.

7. Summary and Concluding Remarks

In this paper we have introduced the orthogonal Hahn moments as pattern features in the analysis of two-dimensional images. Connections between the weight functions of the Hahn, Krawtchouk and Tchebichef polynomials with respect to the weighted image intensity function and limiting cases have been examined. A bound for the truncation error has been derived to assess the accuracy of the image reconstruction. Since the Hahn polynomials include many orthogonal polynomials as limiting cases, the Hahn moments will constitute a general class of orthogonal moments useful in image analysis. Experimental results and local feature extractions by the weighted image intensity function will be discussed in a separate paper.

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