

EXPERIMENTING WITH MAPLE TO OBTAIN SUMS OF BESSEL SERIES

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Abstract

In the study of pulse-width modulation within electrical engineering many authors develop a series representation of the modulated wave using a double Fourier series based on properties of the carrier and reference waveforms. Assuming an integral frequency ratio we can also compute the single Fourier series directly, and equating these we obtain two equivalent representations, one involving the usual trigonometric functions, and the other involving these together with Bessel functions of the first kind. Comparing coefficients we are led to a range of series such as

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(J_{n+1} \left(\frac{\pi n}{2} \right) - J_{n-1} \left(\frac{\pi n}{2} \right) \right) = 1 - \frac{\pi}{4}$$

and these results can be “confirmed” using MAPLE. Generalizing the above series we can then use MAPLE to study the likely sum of each of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(J_{n+p} \left(\frac{\pi n}{s} \right) - J_{n-p} \left(\frac{\pi n}{s} \right) \right)$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(J_{n+p} \left(\frac{\pi n}{s} \right) - J_{n-p} \left(\frac{\pi n}{s} \right) \right)$$

for $p, s = 1, 2, \dots$, all of which converge. In this paper we investigate the patterns that arise, and show that in most cases considered the sums can be given quite specifically and have very simple forms.

This approach is useful as both a research tool and also for introducing students to the rather difficult area of Bessel series.

1 Introduction

The advent of powerful symbolic manipulation packages like MATHEMATICA and MAPLE has had a profound effect on the availability of tools for mathematical research and teaching (for the latter,

see [1] and [2] for example). In this paper we take a more experimental approach, which we illustrate with an example arising from the study of pulse-width modulation within electrical engineering where certain Bessel series (together with their sums) are derived, and then further ones conjectured through use of MAPLE. The principal aim will be to indicate both the general approach and how MAPLE can be used to obtain likely results experimentally, while at the same time avoiding complications in presentation. The format of the paper is as follows. In the second section we derive two seemingly different series representations of a simple periodic function, and in the third section we suggest some derived series that could be usefully investigated and use MAPLE to obtain their likely sums. While these results are developed from just a single example, it is easy to see how the above methods can be adapted to produce a range of other Bessel series.

2 Single and double Fourier series approaches

We consider the reference waveform $v_{ref}(t) = \frac{\pi}{2} \sin t$ and carrier waveform given by

$$v_{car}(t) = \begin{cases} t, & 0 < t < \frac{2\pi}{3} \\ -2t + 2\pi, & \frac{2\pi}{3} < t < \frac{4\pi}{3} \\ t - 2\pi, & \frac{4\pi}{3} < t < 2\pi \end{cases}$$

which have the following graphs:

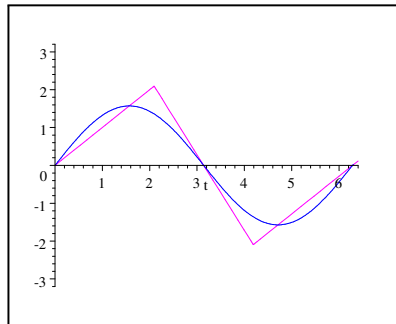


Fig. 1

It is easily checked that the intersection points are, in increasing order, $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$. In the study of pulse-width modulation (PWM) it is usual to consider the modulated (output) wave given by

$$v(t) = \begin{cases} 1, & v_{car} < v_{ref} \\ 0, & v_{car} > v_{ref} \end{cases}$$

We emphasize that while the reference and carrier waves are quite independent, the output wave depends completely on the geometric relationship of these two. We examine two methods for computing

a series representation of the output wave, the first being the single Fourier series of the 2π -periodic function v , and the second derived from an associated double Fourier series.

For the above example, using the first method we are just computing the single Fourier series of the function

$$v(t) = \begin{cases} 1, & 0 < t < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < t < \pi \\ 1, & \pi < t < \frac{3\pi}{2} \\ 0, & \frac{3\pi}{2} < t < 2\pi \end{cases}$$

(the values at the intersection points play no rôle here), which is given by

$$\frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin 2(2n-1)t \quad (1)$$

The two-variable approach involves evaluating the double Fourier series of the doubly 2π -periodic (two-variable) function

$$F(t, y) = \begin{cases} 1, & \pi - \frac{\pi}{4} \sin t < y < 2\pi + \frac{\pi}{2} \sin t \\ 0, & \text{otherwise} \end{cases}$$

and then restricting F to the line $y = t$ (see Fig. 2 below). The enveloping sinusoids are evaluated by solving the equations $v_{ref} = v_{car}$, from which we obtain

$$t = \begin{cases} \frac{\pi}{2} \sin t, & 0 < t < \frac{2\pi}{3} \\ \pi - \frac{\pi}{4} \sin t, & \frac{2\pi}{3} < t < \frac{4\pi}{3} \\ 2\pi + \frac{\pi}{2} \sin t, & \frac{4\pi}{3} < t < 2\pi \end{cases}$$

We are looking at the line $y = t$ and band 0 to 2π . In computing the double Fourier series of the doubly 2π -periodic function F we can take the integration between $\frac{\pi}{2}$ and $\frac{5\pi}{2}$ (shown by dashed lines in Fig. 2), and within this band the boundaries of the region where $F \neq 0$ are from $\pi - \frac{\pi}{4} \sin t$ to $2\pi + \frac{\pi}{2} \sin t$.

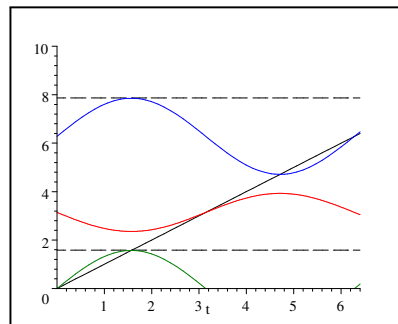


Fig. 2

The required integral is

$$\frac{1}{4\pi^2} \int_0^{2\pi} \int_{\pi - \frac{\pi}{4} \sin t}^{2\pi + \frac{\pi}{2} \sin t} e^{-i(mt+ny)} dy dt$$

which after some manipulation can be shown to be equal to

$$\left\{ \begin{array}{ll} \frac{1}{2}, & m, n = 0 \\ \frac{\mp 3}{16} i, & m = \pm 1, n = 0 \\ 0, & m \neq 0, \pm 1, n = 0 \\ \frac{i}{2n\pi} \left(J_m \left(\frac{\pi n}{2} \right) - J_m \left(\frac{\pi n}{4} \right) \right), & m \text{ even}, n \neq 0, n \text{ even} \\ \frac{-i}{2n\pi} \left(J_m \left(\frac{\pi n}{2} \right) + J_m \left(\frac{\pi n}{4} \right) \right), & m \text{ odd}, n \neq 0, n \text{ even} \\ \frac{i}{2n\pi} \left(J_m \left(\frac{\pi n}{2} \right) + J_m \left(\frac{\pi n}{4} \right) \right), & m \text{ even}, n \neq 0, n \text{ odd} \\ \frac{-i}{2n\pi} \left(J_m \left(\frac{\pi n}{2} \right) - J_m \left(\frac{\pi n}{4} \right) \right), & m \text{ odd}, n \neq 0, n \text{ odd} \end{array} \right.$$

where we appeal to [3], Chapter II for elementary properties of the Bessel functions J of the first kind. Then substituting $y = t$ we obtain

$$\begin{aligned} \sum_{m,n=-\infty}^{\infty} c_{mn} e^{i(m+n)t} &= \frac{1}{2} + \frac{3}{8} \sin t - \\ &- \frac{2}{\pi} \sum_{m=1, m \text{ even}}^{\infty} \sum_{n=1, n \text{ even}}^{\infty} \frac{1}{n} \left(J_m \left(\frac{\pi n}{2} \right) - J_m \left(\frac{\pi n}{4} \right) \right) \cos mt \sin nt + \\ &+ \frac{2}{\pi} \sum_{m=1, m \text{ odd}}^{\infty} \sum_{n=1, n \text{ even}}^{\infty} \frac{1}{n} \left(J_m \left(\frac{\pi n}{2} \right) + J_m \left(\frac{\pi n}{4} \right) \right) \sin mt \cos nt - \\ &- \frac{2}{\pi} \sum_{m=1, m \text{ even}}^{\infty} \sum_{n=1, n \text{ odd}}^{\infty} \frac{1}{n} \left(J_m \left(\frac{\pi n}{2} \right) + J_m \left(\frac{\pi n}{4} \right) \right) \cos mt \sin nt + \\ &+ \frac{2}{\pi} \sum_{m=1, m \text{ odd}}^{\infty} \sum_{n=1, n \text{ odd}}^{\infty} \frac{1}{n} \left(J_m \left(\frac{\pi n}{2} \right) - J_m \left(\frac{\pi n}{4} \right) \right) \sin mt \cos nt + \\ &- \frac{1}{\pi} \sum_{n=1, n \text{ even}}^{\infty} \frac{1}{n} \left(J_0 \left(\frac{\pi n}{2} \right) - J_0 \left(\frac{\pi n}{4} \right) \right) \sin nt - \\ &- \frac{1}{\pi} \sum_{n=1, n \text{ odd}}^{\infty} \frac{1}{n} \left(J_0 \left(\frac{\pi n}{2} \right) + J_0 \left(\frac{\pi n}{4} \right) \right) \sin nt \end{aligned}$$

If we compare this with the single Fourier series (1) then the coefficient of $\sin 2t$ will be $\frac{2}{\pi} = 0.636\ 619\ 772$, and is given (after some rearrangement of the terms) by

$$\frac{1}{\pi} \left(\sum_{n=1}^{\infty} \frac{1}{n} \left(J_{n+2} \left(\frac{\pi n}{2} \right) - J_{n-2} \left(\frac{\pi n}{2} \right) \right) - \sum_{n=1}^{\infty} \frac{1}{n} \left(J_{n+2} \left(\frac{\pi n}{4} \right) - J_{n-2} \left(\frac{\pi n}{4} \right) \right) \right) \quad (2)$$

(which, using MAPLE, evaluates to 0.636 662 541 on taking 1000 terms).

3 Some derived series

For convenience we consider the series (2) without the $\frac{1}{\pi}$ term, and take different values of p as follows:

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(J_{n+p} \left(\frac{\pi n}{2} \right) - J_{n-p} \left(\frac{\pi n}{2} \right) \right) - \sum_{n=1}^{\infty} \frac{1}{n} \left(J_{n+p} \left(\frac{\pi n}{4} \right) - J_{n-p} \left(\frac{\pi n}{4} \right) \right) \quad (3)$$

If we compare this with the sine series (1) then it follows easily that (3) takes the value $\frac{4}{p}$ for $p = 2, 6, 10, \dots$. At this stage it is interesting to see how the series will evaluate for the intermediate values of p . Now in view of the form of (1) the above analysis will not give the sums for values of p other than 1 and $2(2r - 1)$ where $r = 1, 2, \dots$. However we can compute these using MAPLE, which we do to respectively 200, 500 and 1000 terms, to obtain the table below. For example, the MAPLE code that produces the fourth column (other than $2(2r - 1)$ for $r = 2, 3, \dots$) is

```
Sum('(BesselJ(n+p,Pi*n/2))/n',n'=1..500)-Sum('(BesselJ(n-p,Pi*n/2))/n',n'=1..500)-
-Sum('(BesselJ(n+p,Pi*n/4))/n',n'=1..500)+Sum('(BesselJ(n-p,Pi*n/4))/n',n'=1..500);
for p from 3 by 1 to 25 do print(p,evalf(%)) od;
```

p	1/p	200 terms	500 terms	1000 terms
3	0.3333333333	0.6673280381	0.6666251115	0.6667322114
4	0.2500000000	-0.0005241617	0.0000299425	-0.0000508246
5	0.2000000000	0.3986547897	0.4000760534	0.3998698643
7	0.1428571429	0.2855500932	0.2857246986	0.2856981155
8	0.1250000000	0.0010127370	-0.0000453193	0.0000939872
9	0.1111111111	0.2236969817	0.2221598011	0.2223578543
11	0.0909090909	0.1814216279	0.1818282126	0.1817831158
12	0.0833333333	-0.00139547339	0.00003703587	-0.00012295589
13	0.07692307692	0.1524538578	0.1538783341	0.1537248681
15	0.06666666667	0.1342644115	0.1333269983	0.1334136004
16	0.06250000000	0.00156332743	-0.00000566990	0.00013336322
17	0.05882352941	0.1187105450	0.1176469335	0.1177365729
19	0.05263157895	0.1039820734	0.1052363911	0.1051507732
20	0.05000000000	-0.00141813482	-0.00003690608	-0.00012382342
21	0.04761904762	0.09468667594	0.09522435780	0.09519240924
23	0.04347826087	0.08822449498	0.08703877668	0.08708298648
24	0.04166666667	0.00096051646	0.00006963995	0.00009637587
25	0.04000000000	0.08005416230	0.07999121369	0.07999739795

It turns out that the convergence of these series is rather slow which makes the computations a little tedious. On examination of the above table of values (and including the results already obtained analytically) it is apparent that the pattern is

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(J_{n+p} \left(\frac{\pi n}{2} \right) - J_{n-p} \left(\frac{\pi n}{2} \right) \right) - \sum_{n=1}^{\infty} \frac{1}{n} \left(J_{n+p} \left(\frac{\pi n}{4} \right) - J_{n-p} \left(\frac{\pi n}{4} \right) \right) = \begin{cases} 2 - \frac{\pi}{8}, & p = 1 \\ \frac{2}{p}, & p = 2r - 1, r = 2, 3, \dots \\ \frac{4}{p}, & p = 2(2r - 1), r = 1, 2, \dots \\ 0, & p = 4r \end{cases} \quad (4)$$

Now by considering an example with a different choice of reference and carrier waveforms, respectively $v_{ref}(t) = \pi \sin t$ and the triangular wave

$$v_{car}(t) = \begin{cases} -2t, & 0 < t < \frac{\pi}{2} \\ 2t - 2\pi, & \frac{\pi}{2} < t < \frac{3\pi}{2} \\ -2t + 4\pi, & \frac{3\pi}{2} < t < 2\pi \end{cases}$$

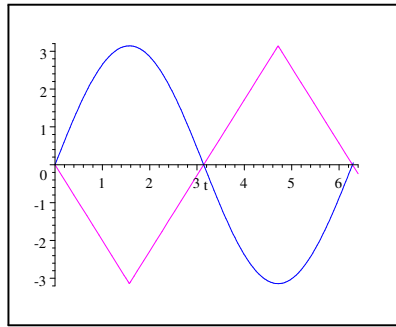


Fig. 3

we obtain the output wave

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(J_{n+p} \left(\frac{\pi n}{2} \right) - J_{n-p} \left(\frac{\pi n}{2} \right) \right) = \begin{cases} 1 - \frac{\pi}{4}, & p = 1 \\ \frac{(-1)^{p+1}}{p}, & p \geq 2 \end{cases} \quad (5)$$

If we translate the carrier by π to the right then the graph becomes

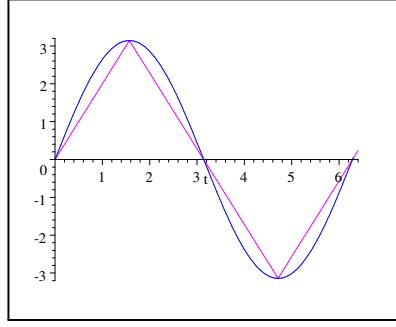


Fig. 4

and this gives the output wave

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(J_{n+p} \left(\frac{\pi n}{2} \right) - J_{n-p} \left(\frac{\pi n}{2} \right) \right) = \begin{cases} 1 - \frac{\pi}{4}, & p = 1 \\ \frac{3}{p}, & p = 2(2r - 1), r = 1, 2, \dots \\ -\frac{1}{p}, & p = 4r \\ \frac{1}{p}, & p = 2r - 1, r = 2, 3, \dots \end{cases} \quad (6)$$

Again some of the values have been obtained analytically, and the remaining ones either using MAPLE or comparing with those obtained earlier.

We can consider the following series related to that in (4):

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{n} \left(J_{n+p} \left(\frac{\pi n}{2} \right) - J_{n-p} \left(\frac{\pi n}{2} \right) \right) + \sum_{n=1}^{\infty} \frac{1}{n} \left(J_{n+p} \left(\frac{\pi n}{4} \right) - J_{n-p} \left(\frac{\pi n}{4} \right) \right) \\ & = \begin{cases} -\frac{3\pi}{8}, & p = 1 \\ -\frac{2(-1)^{p/2}}{p} & p \text{ even} \\ 0, & p \text{ odd} \end{cases} = \begin{cases} -\frac{3\pi}{8}, & p = 1 \\ \frac{2}{p}, & p = 2(2r - 1) \\ -\frac{2}{p}, & p = 4r \\ 0, & p = 2r - 1 \end{cases} \end{aligned} \quad (7)$$

If we subtract (4) from (7) we have

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(J_{n+p} \left(\frac{\pi n}{4} \right) - J_{n-p} \left(\frac{\pi n}{4} \right) \right) = \begin{cases} -1 - \frac{\pi}{8}, & p = 1 \\ -\frac{1}{p}, & p \geq 2 \end{cases} \quad (8)$$

Now consider an alternating version of the original series (4) then (we expect from other analysis

to obtain 0 for the even terms)

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(J_{n+p} \left(\frac{\pi n}{2} \right) - J_{n-p} \left(\frac{\pi n}{2} \right) \right) - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(J_{n+p} \left(\frac{\pi n}{4} \right) - J_{n-p} \left(\frac{\pi n}{4} \right) \right) \\ &= \begin{cases} \frac{\pi}{8}, & p = 1 \\ 0, & p \geq 2 \end{cases} \end{aligned} \quad (9)$$

Thus the original series leads to a variety of results involving Bessel series. We have carried out the calculations to 500 terms. The loss of accuracy compared with taking 1000 terms is offset by the considerable difference in computational time (around 8 minutes compared with over 2 hours for each p -value). Note that from (5) and (9)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(J_{n+p} \left(\frac{\pi n}{4} \right) - J_{n-p} \left(\frac{\pi n}{4} \right) \right) = \begin{cases} -1 + \frac{\pi}{8}, & p = 1 \\ \frac{(-1)^p}{p}, & p \geq 2 \end{cases} \quad (10)$$

For this to work we must have comparing (8) and (10)

$$\sum_{n=1, n \text{ odd}}^{\infty} \frac{1}{n} \left(J_{n+p} \left(\frac{\pi n}{4} \right) - J_{n-p} \left(\frac{\pi n}{4} \right) \right) = \begin{cases} -\frac{1}{p}, & p \text{ odd} \\ 0, & p \text{ even} \end{cases}$$

which gives

$$\sum_{n=1}^{\infty} \frac{1}{2n-1} \left(J_{2n-1+2r} \left(\frac{\pi(2n-1)}{4} \right) - J_{2n-1-2r} \left(\frac{\pi(2n-1)}{4} \right) \right) = 0, \quad r = 1, 2, \dots$$

and then considering the even terms

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(J_{2n+2r} \left(\frac{\pi n}{2} \right) - J_{2n-2r} \left(\frac{\pi n}{2} \right) \right) = -\frac{1}{r}, \quad r = 1, 2, \dots$$

After scaling the arguments of the Bessel functions we also obtain

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(J_{2n+2r} \left(\frac{\pi n}{4} \right) - J_{2n-2r} \left(\frac{\pi n}{4} \right) \right) = -\frac{1}{r}, \quad r = 1, 2, \dots$$

3.1 Related MAPLE calculations

Let us write

$$B(p, s) = \sum_{n=1}^{\infty} \frac{1}{2n-1} \left(J_{2n-1+p} \left(\frac{\pi(2n-1)}{s} \right) - J_{2n-1-p} \left(\frac{\pi(2n-1)}{s} \right) \right)$$

As we have indicated above,

$$B(2r, 4) = 0, \quad r = 1, 2, \dots$$

We carry out some MAPLE calculations on $B(2r, s)$ to obtain

$$B(p, s) = \begin{cases} 0, & p \text{ even}, s = 4, 5, \dots \\ -\frac{1}{p}, & p \text{ odd}, s = 4, 5, \dots \end{cases}$$

and then to evaluate $B(p, s)$, $s = 1, 2, 3$ we use MAPLE (taking 500 terms) to obtain the second column in the table below ($s = 1$). The third and fourth columns are obtained by taking respectively $s = 2$ and $s = 3$. Again, the MAPLE code that produces the first twenty terms of the third column is just

```
Sum('BesselJ(2*n-1+p,Pi*(2*n-1)/2))/(2*n-1)',n'=1..500)-
-Sum('BesselJ(2*n-1-p,Pi*(2*n-1)/2))/(2*n-1)',n'=1..500);
for p from 1 by 1 to 20 do print(p,evalf(%)) od;
```

p	s = 1	s = 2	s = 3
1	0.6764670274	0.0000544408	-0.8651447921
2	0.5424173346	1.000069315	0.2516887675
3	-0.2637429485	0.0000338149	0.0023566449
4	0.4963901368	-0.0000262565	0.3778255811
5	-0.1084479066	-0.00006723912	0.1762554241
6	0.1246764938	0.333273981	0.3363362761
7	0.1261785257	-0.00000834001	0.1263936354
8	0.00713549307	0.000048712	0.1895833742
9	0.004358859743	0.00007034242	0.00126357391
10	0.1411946588	0.200040856	0.05027583901
11	-0.08632944432	-0.00001828075	-0.07953113691
12	0.1560904728	-0.00006407171	-0.00187671231
13	-0.01779874452	-0.0000632691	-0.06946849184
14	0.03141182811	0.1428406105	0.03208594635
15	0.06593104558	0.00004210202	-0.00412303529
16	0.01388883322	0.00007002747	0.0894793954
17	0.00374667871	0.00004705953	0.04690746575
18	0.09399528807	0.1111011580	0.1076953129
19	-0.05263309027	-0.0000594978	0.04313128755
20	0.08302627901	-0.00006566717	0.07379197358
22	0.00891567066	0.09094355957	0.02438331222
24	0.01981220805	0.00005174253	0.00415294772
26	0.07281944416	0.07687001116	0.02457754502
28	0.04901978572	-0.00003070411	0.05838982952
30	0.00145516236	0.06672938729	0.06955502203
32	0.02468399352	0.0000064179410	0.04735578484
34	0.05857185053	0.05876149111	0.01335770433
36	0.02896159187	0.00001647671	-0.00199277895
38	0.00002031819	0.05268331663	0.01149825593
40	0.02817625471	-0.00003352029	0.03624103299

The pattern for $B(p, 2)$ is clear, and in fact we have obtained experimentally

$$B(p, s) = \begin{cases} \frac{2}{p}, & p = 2(2r - 1), r = 1, 2, \dots, s = 2 \\ 0, & p \neq 2(2r - 1), r = 1, 2, \dots, s = 2 \\ 0, & p = 12r, r = 1, 2, \dots, s = 3 \\ 0, & p \text{ even}, s = 4, 5, \dots \\ -\frac{1}{p}, & p \text{ odd}, s = 4, 5, \dots \end{cases}$$

It is not at all obvious from the above table what the general expressions for $B(p, 1)$ and $B(p, 3)$ would be but, for example, we can show that for p odd

$$B(p, 3) = \frac{\cos \frac{5p\pi}{6}}{p}$$

which gives $B(1, 3) = -\frac{\sqrt{3}}{2}$, $B(3, 3) = 0$, $B(5, 3) = \frac{\sqrt{3}}{10}$, $B(7, 3) = \frac{\sqrt{3}}{14}$, $B(9, 3) = 0, \dots$

4 Conclusion

The above approach leads to a variety of Bessel series and their sums, many of which have been verified directly by comparing single and double Fourier series.

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