Teaching Derivative with Graphic Calculators: The Role of a Representational Perspective

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Abstract

There is now a considerable literature on the value of an approach to learning mathematics which incorporates multiple representations of concepts. A widespread application of this approach will require the full support and appreciation of teachers, along with suitable professional development. However, it is not entirely clear yet how assisting classroom teachers to build the theoretical ideas into successful teaching approaches relates to their content knowledge and pedagogical content knowledge. In our current research project we are working with a group of teachers using graphic and CAS calculators in the classroom teaching of processes and concepts associated with derivative. This paper describes the way in which interviews and classroom observation confirm how the representation-based content knowledge and pedagogical content knowledge of derivative of one teacher influenced her use of the calculator and her teaching of derivative.

Background

The complexity of teacher knowledge and its relevance to teacher practice has been the object of a number of research projects in mathematics education in recent years (e.g., Bromme, 1994; Cooney, 1999; Simon & Tzur, 1999). However, while most would agree that it is important that teachers should have a sound understanding of the mathematics they teach, mathematical content knowledge alone does not automatically translate to favourable outcomes (Bromme, 1994; Cooney, 1999). In fact Bromme's (1994) review of research investigating the relationship between teacher content knowledge and student performance, highlighted little evidence of a link, and even concluded that "a relatively low stock of knowledge is sufficient to teach students" (p. 76).

Two other factors influencing teachers' efforts to design experiences to support students' construction of rich conceptual thinking are their pedagogical content knowledge and their knowledge of how students understand (Shulman, 1986, Simon, 1995; Chinnappan and Thomas, 2001). Pedagogical content knowledge, in the sense of Shulman (1986), refers to an understanding of what it means to teach a particular topic, as well as knowledge of the principles and techniques required to do so. Describing its emergence, Wilson, Shulman, and Richert (1987, p. 114) talk of how, as teachers "prepare to teach their content as well as during actual instruction, they develop a new type of subject matter knowledge that is enriched and enhanced by other types of knowledge". Chinnappan and Thomas (2001, p. 115) suggest that it includes "information that the teachers hold about students' understanding, such as their conceptual and procedural knowledge" of the mathematical domain to be taught. Summarising, Simon (1995, p. 133) succinctly puts it this way,

> Beside the teacher's knowledge of mathematics and his hypothesis about students' understanding, several areas of teacher knowledge come into play, including the teacher's theories about mathematics teaching and learning' knowledge of learning with respect to particular mathematical content and knowledge of mathematical representations, materials and activities. (Simon, 1995, p. 133)

An integral part of teacher pedagogical content knowledge, Simon maintains, is teachers' representation-based knowledge. Mathematical symbols represent and embody meanings of mathematical objects as a result of their association with conceptual structures (Lerman, 1994). Hence the teacher can employ symbol use to enrich student experiences and mathematical understanding by actively engaging them in cognitive processes. In turn, symbols and symbolic representational systems provide a means by which mathematical structures are represented (Kaput, 1998). However, while it is important to interact with each representational system in a versatile way, both procedurally and conceptually (Thomas, 2001, Thomas & Hong, 2001), a single representational system cannot capture the full meaning of a mathematical construct. Instead mathematical meaning is distributed across several interacting representational systems, with each emphasising and de-emphasising different characteristics of the construct (Lesh, 1999). Hence there is a need for teachers to make explicit opportunities for students to investigate links between representations of a given concept, to assist them to construct representational fluency (Lesh, 2000).

Apart from these different facets of teacher knowledge, another important factor influencing the teaching-learning process is a recognition, by the teacher and the students, that their relationship has *reciprocal obligations*, that it resembles a social contract (Brousseau, 1997). This *didactic contract* as it has been named (Brousseau, 1986) is a dynamic defining of the role that both the teacher and the students play in the classroom. While it may be explicit or implicit, formal or informal, it becomes binding as each participant reacts to, or accepts, situations that result during the classroom discourse. For example, teachers are required to provide create the environment and the mood in the classroom for the appropriation of knowledge and to be sensitive to when this occurs (Brousseau, 1997), while students are expected to learn.

There is some evidence that the use of technological tools, such as computers and graphic calculators, when made an integral part of a didactic contract, can provide valuable support for a multiple representational approach to teaching. In this sense they act as a physical medium for the purpose of re-expression and translation (re-presentation), and for mathematical communication (Noss & Hoyles, 1996; Yackel, 2000). However, when teachers use graphic calculators they may stress different teaching goals, since these are anchored in their philosophies of mathematics and mathematics education and their beliefs about the potential of the calculator to support them (Doerr & Zangor, 1999; Simmt, 1997; Harskamp, Suhre, & van Streun, 1998). Such goals can include student-centred initiatives, interactive-inquiry teaching styles and student-centred views of learning, as well as content-oriented goals and a view of learning as listening. Kendal and Stacey's (1999) research with teachers who taught differential calculus using computer algebra system (CAS) calculators found that the students learning in class was influenced by *teacher privileging*. Student preferences mirrored aspects of learning that teachers gave greater emphasis to, in line with their own personal teaching styles and attitudes. This didactic contract effect exemplifies the key role of teachers in their implementation of GC use.

This paper considers the nature of one experienced teacher's multiple representational approach to the teaching of derivative using integration of the graphic calculator in her classroom. We were interested in examining features of her didactic contract, as well as her ideas about mathematics and how it should be taught, and the role of representations. We investigated how her representation-based content knowledge and pedagogical content knowledge of derivative influences her use of the calculator and her teaching of derivative. We particularly wanted to address the following questions:

- Can we describe aspects of the nature of the didactic contract for this teacher?
- How do the teacher's classroom practices relate to her own understanding of mathematics and her pedagogical content knowledge in the context of graphic calculator support of a multiple-representational approach to teaching?

Method

In this study, which forms part of a larger study with four teachers on differentiation and derivative, we have applied a methodology similar to that which Simon and Tzur (1999) introduced in their effort to explain the teacher's perspective from the researcher's perspective. We have attempted to understand the coherence of one teacher's practice in relation to her pedagogical content knowledge, her representation-based content knowledge, and her beliefs about how the graphic calculator can be of support in enhancing the teaching-learning environment.

Teacher Interview Script (Selected questions only)

1. **The teaching of derivatives and differentiation**
   - Have you taught derivatives before?
   - What is your first lesson on derivatives and differentiation? What is your follow up lesson?
   - Could you tell me about the factors that you consider important when planning for your lessons?
   - Could you give a brief outline of how you have gone about teaching derivatives?
   - What related ideas do you use in introducing derivatives to your students? Why have you considered them important?
   - What kind of examples do you use in teaching derivatives?
2. **The learning of derivatives and differentiation**
   - What do you think students need to know about derivatives and differentiation?
   - What key ideas should they learn?
   - What skills should they master?
   - What ideas do students need to know before studying derivatives?
   - How do you think students learn or go through the process of mathematical understanding of derivatives and differentiation?
   - What factors contribute to better learning and mathematical understanding of students?
   - What should the emphasis in the teaching of derivatives that will help the students to better understand?
   - How will you measure students' success in learning derivatives and differentiation?

Figure 1. Sample questions from the teacher interviews.

Our respondent is an experienced female teacher working is a co-educational secondary school of high academic status in Auckland, New Zealand. We will refer to her using the pseudonym Rachel. The class referred to below consists of 20 Year 13 students (aged 17 to 18) who have chosen to take Mathematics with Calculus as an elective subject in preparation for entry to university. A key criterion for choosing both the school and the teacher (and her students) was a willingness to use graphic calculators in teaching (in fact Rachel has already been using graphic calculators in her teaching for more than 5 years). Data was collected in three stages. In the initial stage, Rachel attended a workshop (along with the 3 other teachers involved in the larger project) to discuss ways by which derivative could be taught with the use of graphic calculator. After this she was interviewed for the first time about the teaching and learning of differentiation and derivative (see Figure 1). Next the first–named researcher observed her classes and constructed detailed field notes to describe her manner of teaching, including her implementation of graphic calculators in the classroom. A second interview was conducted after Rachel had completed her course on derivative. In this second interview she was encouraged to reflect on her teaching practice. To assist in validating the researchers' interpretation made in this paper of the teacher's perspective, the first draft of the paper was shown to the teacher for comment.

Based on the responses of the teacher in the interview, we attempt to map out the typology of the teacher's knowledge in our effort to understand it as one of the sources that informs teacher practice. The field notes of the first author during classroom observation have been interpreted for evidence establishing a correlation between the teacher's knowledge (collected from pre- and post-teaching interviews) and her teaching practice. We understand that the results obtained from this study will not generalise to all teaching of derivative with graphic calculators. However, they may provide useful information and be a source to further questions for research in this area.

Discussion

The Didactic Contract

In elucidating the idea of the didactic contract, Brousseau (1986) focuses our attention on what the teacher should teach and how, and what the student should know and how she should come to know it. Rachel has very clear ideas about what her role is in the didactic contract, especially in her classroom where GCs are given a strong emphasis. She espouses the importance of a student-centred approach, building on problems which generate student discussion in the classroom, although she states, "I don't suppose I'm following any sort of theoretical principle, but I just have found that I want to be able to engage students". This student-centred view is seen, for example, in her first interview, where she said:

> My lessons involve a lot of discussion…and in terms of engaging students, I like to involve them as much as I can in questioning…they need to be there with me all the time…mentally. My classes…they're very much discussion-based. It's quite informal, I encourage interactions, within limits. I encourage students to see me if they've got misunderstanding of what's going on…there's a lot of discussions about problems, and the problems that I choose, they're put on my worksheets. I choose them quite carefully, and hopefully each problem must show another aspect of what I'm trying to get at. They're not …a whole lot of routine examples. I don't see them as examples. I suppose I see them as discussion items.

These comments also make clear some of her expectations of students. In line with Brousseau's (1997) theoretical position, hers is a classroom with an undercurrent of discipline, and there are clear 'limits' set for student interactions; they are encouraged to ask about misunderstandings; have to work on a set list of problems; they "have to be with me all the time", etc. While she is clearly the teacher, she makes a point of having her own learning as part of the contract. She says:

> So then in my learning … if a student shows me something, I always try immediately show others … I think its really important that we are all learners and we never ever stop learning and I always say to them, gosh, I learnt so much from my students and they often feed me stuff.

This openness to learning in the didactic contract was demonstrated in her classroom practice too. In the second lesson she noticed that that one of the students was using a split screen to view the graph and the home screen together. Rachel asked this student to show the others how to split the screen and she then demonstrated the procedure based on instructions from the student, commenting that the split screen could give a view that relates symbolic and graphical representations. Following this incident there was an air of excitement from the students in the room as they too "split screens." Also illustrating her attitude to learning is her description of how she has had to move out of her 'comfort zone' on occasion to work alongside the students. One area was in the use of tables on the GCs:

> I don't feel comfortable in tables. Maybe I should push myself out of my comfort zone and just see what I can get out of the tables, … I was so surprised that they were very comfortable with tables and they would often flip to the table and it's a challenge for me because I wasn't as comfortable in them, but hopefully I'll be more comfortable and hopefully I'll use it more and hopefully I'll use it in my classroom and hopefully it becomes second nature.

She believes that GCs are a means of accessing a rich vein of learning, if they are used well.

> It is actually crucial because you're not only learning to use a calculator but you're learning how to learn… I'll

talk about students who just reproduce graphs and I often think gosh if only you realised that that is only that's where get their names from because they can draw graphs but there's just so much rich mathematics in that machine that you can release if you can. I'm not an expert but there's so much mathematics you can release. The more you get to know about the machines the more you can work on it.

Importantly students have to form a partnership with their calculator, so they have to be available and used constantly: "you got to have them in your hands all the time just got to have them its got to be a part of them." Enlarging on this last point she explains how she would like to see the students' thinking and understanding influenced by the GCs, without them necessarily being aware of this, and tells the example of a student who had thought through a problem by visualising a graphical translation:

She was the one that had said 'just a vertical translation graph' and I thought 'isn't that interesting she doesn't think that she's necessarily using the skills that she's learnt from using the graphic calculator, but she is using them without even realising she's using them.

However, she is realistic about GC limitations too: "I still don't like the graphs coming up with the pixels … there's lots of limitations with graphic calculators". For students to form a working partnership with the calculator, they must think about the underlying mathematics and relate the screen display to this:

I've realised its one of the most important things about the graphic calculator. What you see is not necessarily what you get but they've got to actually think about.. what are you expecting to see, what adjustments to make … This mathematical understanding we want to develop into the students and the use of the graph calculator they go together,.. develop together.

One way to achieve this, and a clear part of the perceived didactic contract for Rachel, is her privileging of the use of the GCs, in the sense of Kendal and Stacey (1999), by acting as a role model; "if you model the use of it they feel comfortable with it". In practice Rachel's pedagogical approach was seen to differ somewhat at times from what she described in her interviews. Her lessons tended to follow a standard procedure, as she acknowledges:

I suppose I have a fairly traditional lesson plan. I have to go through a couple of cycles in 85-minute period, and there will be follow up to consolidate what I've done, and obviously at the beginning of the next period, just going around checking. I would start a period by, perhaps, just could be checking the homework and seeing if there's any problem in the previous lesson. And if there's something from the previous lesson that I want to use, to build up, to start the next lesson. Well, obviously, I want to start with that as an introduction.

So she begins by reviewing the previous lesson, identifying any problems in understanding. This was followed by a discussion, but one which was clearly teacher-led, rather than a problem-centred student discussion. Then students would work on a problem, often in pairs, when they would engage in discussion. During this time Rachel moved around the room supporting the students in their work and assisting the student pairs where necessary. In the second lesson on derivative we see an example of where she succumbed to telling students rather than getting them to find a solution. The students were given the example of the function $f(x)=x^3$, with the task of considering its derivative. Using the values of the function for x=1, 2, 3 and 4 and finding the first and second differences, a table showing the following values was constructed:

Derivative Value	1st Difference	2nd Difference
3		
12	9	
27	15	6
48	21	6

This would seem to be an ideal opportunity to get the students to engage in investigation and discussion, but instead Rachel told the students to note that the 2nd difference is a constant and said "so we expect a quadratic."

Lying behind this difference between Rachel's pedagogical content knowledge and her classroom practice are many constraints impinging on her classroom preparation (Wilson, *et al.*, 1987), which could be described as *external conditions* in terms of the didactic contract. One major one which most teachers will relate to is the looming external assessment at the end of the year. As she says, "I'm preparing the students for Bursary [year 13 university entrance examination], so I've got to be aware that they're going to.., if they're going to use this calculator, that they've got to be able to explain what they're doing." It means that there is also a certain amount of work that needs to be covered. She is conscious of this crucial time restraint, "I would just like to have more time to explore for the students to discover things for themselves" and "there's just so much you could do to help the students to better understanding rather than,.. yes, that's what I'd like to do, I would like to have more time to better concepts".

Linking Graphs and Algebra with GCs

When it comes to the practice of teaching specific constructs of derivative there are a few ideas that Rachel has at the forefront of her thinking. The basic concept for her is the same as that expressed by the teacher Margot in the work of Chinnappan & Thomas (1999), namely:

> I'm constantly enphasising the change of one variable with respect another and that's the essence of it all… You're looking at a function and this is constantly changing in terms of how one variable changes and the rate of change is changing depending on the way the independent variable is changing.

A second fundamental part of her teaching of derivative is that of emphasising a visual approach to learning, supplementing an algebraic one:

> I suppose that's my visual thing again. I enjoy that curve sketching and I suppose it comes back to maths modelling. If you're going to model something using some sort of algebraic method if you can imagine that as a curve you can imagine what's going on there, then that must surely enhance your understanding of what's going on.

This visual aspect Rachel sees as facilitated by her promotion of a multiple representation approach to learning, as recommended by Lesh (1999).

> I always like to try and be linking the graphical representation again by looking at symbolic representation. By this stage, they've probably got the idea of how to basically go through differentiation and then have to do it the procedural way. But then when you go back and tie up with the graph… that whole idea of linking it to it…

Her use of the representations is also based on the broader premise that multiple approaches are beneficial: "I always think that if you can get a concept across and explain it in as many different manners, as many different ways as possible, then the students get the best chance of comprehending." The GC has a pivotal role to play in a multiple representation approach for Rachel, as can be seen from one of her responses.

> Int: And in what ways do you use graphic calculators?
>
> R: I tried to use them in multiple representation, that's what I tried to do, I tried all the time. … I know that multiple.., thinking flexibly, all that sort of thing but I just like to have another learning approach, I just think that students learning different ways approaching a problem, it must enhance understanding.

Rachel described an example in her interview to illustrate the kind of beneficial linking between the algebraic and graphical modes that she is aiming to help her students achieve. She says:

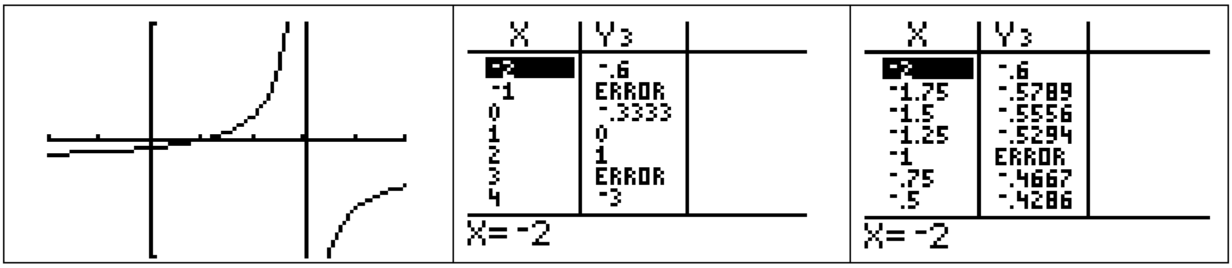
> They had defined the value of p such the quartic had three stationary points, and when you differentiate it you get the q back, which factorises, so you knew straight away that one of your stationary points was zero, and

then of course you were left with this quadratic and I think this quadratic is something like $4x^2 + p$. So the factorisation of the derivative was $2x(4x^2 + p)$, that's right, about the other factors so you had $2x(4x^2+p)=0$ at the stationary point. So they knew that one stationary point was $x=0$ and then they had to think about what value of p would give them two real roots. Ok, you can do that algebraically, but I said to them, what do you see, if I say to you $4x^2$ what do you see? I often say this to my class. Can you imagine the graph? Can you see it? And then they nod, and then I say well can you see the actual factors or whatever? So I say to them, can you see $4x^2$?' Yeah, I can see. What does it look like? Well, it's a parabola. Well, Ok, now what does the $+p$ at the end mean? A couple of times, well that's a vertical translation. Ok, now where would the value of p have to be if you have two real roots? And that you will have to shift the curve down and p has to be negative. … Ok, now let's look at this from an algebraic perspective. How would you do it algebraically?

Here in the interview she is very clear on the didactic value of linking the graph with the algebra. This also translates into action in her lessons on derivative.

Linking Graphs and Tables with GCs

Rachel took a more traditional route in the teaching of derivative than many today would espouse. Rather than encouraging students to see graphs as 'locally linear' where the functions are differentiable, after the manner first described by Tall (1985), she introduced the topic by a consideration of limits of gradients of chords of graphs. In her first lesson, after preliminaries on the concept of limit, she considered some specific examples. One of these was $\lim_{x\to -1}\dfrac{1-x^2}{x^2-2x-3}$. In discussing this she first factorised the denominator, and leaving the numerator blank, and writing $\lim_{x\to -1}\dfrac{\quad}{(x+1)(x-3)}$ she asked the class for any asymptotes of the function. The students responded x at –1 and 3. Rachel then graphed the function using the GC showing a parabola (see Figure 2a), and indicated that there is an asymptote at 3 but not at –1. She then asked the students why there is no asymptote at –1. One student answered that the factor $(x+1)$ will be cancelled, while another answered that there is a "hole" in the graph at $x=-1$. This latter student referred to the display on the screen of the calculator, and suggested that if you zoom in then "then there's a hole in the graph."



a b c

Figure 2. A multiple representation approach to the limit of the function $f(x)=\dfrac{1-x^2}{x^2-2x-3}$ as $x\to -1$.

The teacher then factorised the numerator to show the result of cancelling what she called the "offending factor." and followed this with an explanation of the presence of the factor. It was in order to make this conceptual point clear however, that Rachel resorted to another representation. She used the GC to get the students to consider the tabular values where they could see that even though there is no value at $x=-1$, there is a value being approached (see Figure 2b & 2c). In this way the students were interacting with the algebraic, graphical and tabular representations in a conceptual manner (Thomas & Hong, 2001), in order to understand the effect of a repeated factor in a function. Such an approach also points out a limitation of the GC technology whereby the 'hole' or discontinuity in the graph is not visible in the graphing mode but is in the table, and hence motivates the need to interpret the graph it produces in the light of the algebraic and tabular representations.

Linking Graphs, Algebra and Tables with GCs

In her second lesson on derivative Rachel was introducing the idea that the gradient at a point on a curve is the gradient of the tangent at that point. Implementing the graphic calculator, she drew the graph of the equation $y=x^2$ on it. Then she drew the tangent line at x=2 (the TI-83 shows the equation of the tangent line in the lower left corner of the display – see Figure 3a), and asked:

R: What is the gradient of the curve at x=2?

Is it the same as the gradient of the tangent? [at x=2]

When x=1, the gradient of tangent = 2.

Therefore, the gradient of tangent at 1 is 2.

Meanwhile she was drawing the tangent at x=1 on the GC screen (Figure 3b) and the students were busy following on their GCs and checking their screens against hers.

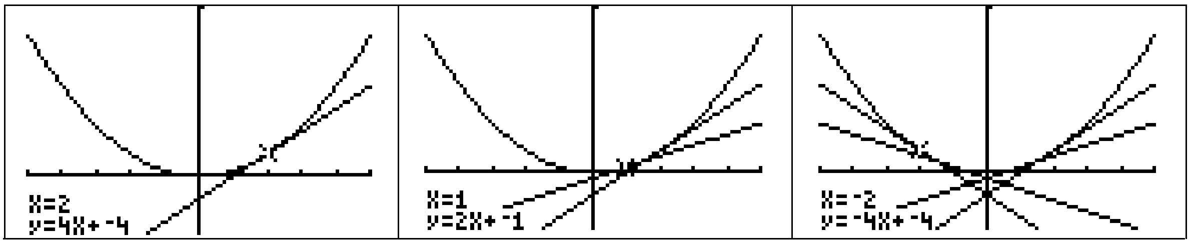


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Figure 3. $y=x^2$ and two if its tangents drawn on the Ti-83.

This was followed by drawing the tangents at the further points, x=0, –1, –2 (see Figure 3c). Rachel then asked the class to guess the gradient at x=–2. She received two responses, the first of which showed considerable conceptual insight:

Student 1: Because of the symmetry.

Student 2: Because when you derive $y=x^2$, then $f'(x)=2x$. And so, when x=-2 the $f'(x)$=-4.

Teacher: Very good [repeats the responses given by both students, accepting both as reasonable]

The versatility of thought (Thomas, 2001, 2002) shown here by the student in using symmetry rather than simply grinding out the answer procedurally, as the second student did, is an example of what Rachel is hoping to achieve by her teaching approach. In this episode, it can be seen that the teacher has used the GC to establish a pattern for students to interact with, and has made a link between gradient of tangent at a point and the gradient of a point on a curve. She continued to stress the multiple representational aspects in this lesson by introducing tables on the GC.

X	Y1	Y2
0	0	0
1	1	2
2	4	4
3	9	6
4	16	8
5	25	10
6	36	12

X=0

Figure 4. Use of the GC table representation for finding the derivative of the function $y=x^2$.

She first demonstrated the use of the TI-83's nDeriv( ) function and then worked with the example, $Y_1=x^2$, $Y_2=$nDeriv(Y_1,X,X), explaining to the students what each of the parameters in the nDeriv( ) function meant as she was inputting them. She discussed the tabular pattern she obtained for the functions (see Figure 4), asking the students for any pattern they observed and its algebraic rule. Eventually, based on the responses coming from the students she was able to write $Y_2=2X$ on the board.

Conclusions

Mathematical discourse in the classroom is clearly a major influence on students' conceptual development and mathematical understanding. The direction of this discourse can be characterised by various factors such as the background of the teacher and the students, the teacher-designed activities, the conduct of classroom interaction, the artefacts used by the teacher and the students, and the presentation and representation of mathematical objects, among others. The orchestration of the complex interrelationship of the above-mentioned factors when forming a didactic contract is not an easy task. In particular many teachers do not find it easy to integrate the use of technology such as the GC into their lessons. They may be averse to doing so, or if they do introduce it then they are limited to a purely procedural use of the calculator (Thomas & Hong, 2001). This was not the case with Rachel, and it appears that conceptual use of the GC in a multi-representational way may be intrinsically dependent on teacher variables. It is not easy to evaluate what the crucial factors are, but some principles which have emerged from our study which we believe may be useful in helping to produce didactic contract with a positive atmosphere for learning with GCs include:

- The personal attitudes and perspectives of a teacher will be a driving force behind the direction of her teaching. If the teacher is not open to certain approaches then these will not eventuate.
- It is necessary to have well thought out and prepared problems which illustrate points of learning in order to construct a positive, student-centred classroom environment.
- The ability to recognise constraints on what a teacher may consider the ideal learning situation and to be prepared to work within these.
- The teacher is prepared to model the behaviour she wants students to emulate. For example having the GC to hand and using it constantly.
- The teacher has a willingness to be open to her own learning in the classroom, even if this requires her to go outside of her own 'comfort zone'.
- The teacher can reflect on the tension created between valuing a formal, primarily algebraic approach to mathematics and an investigative style of teaching with the GC, can contribute to sustaining a central route.
- The teacher's reflection on her own learning experiences in comparison with her preferred learning style may be helpful in resolving teaching preferences.

In Rachel's case we have seen that her practice has initially been influenced by her strong personal belief in a multi-representational and multi-strand approach to learning which engages students. She found a support for this pedagogical stance in the GC, and her flexibility and willingness to learn alongside her students has enabled her to construct classroom practice based on a didactic contract which she finds both satisfying and fulfilling. We hope that this brief analysis of her pedagogical perspectives will prove useful in a consideration of other teacher practice.

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