LOCKWOOD'S "CURVES" ON A GRAPHICS CALCULATOR

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Abstract

Many teachers and students of secondary mathematics in the 1960s and 1970s have fond memories of E. H. Lockwood's *A Book of Curves*. It was a wonderful compendium of information about two-dimensional curves both familiar and strange, with a mixture of geometry, co-ordinate geometry and calculus illuminating both theory and exploration. The drawing exercises in particular were a great source of inspiration to many — teachers and students — looking for "enrichment" activities.

In this paper I take some of Lockwood's ideas, concentrating on his first chapter on the parabola, and examine to what extent they can be reworked to suit the abilities and interests of today's students, and indeed enhanced, using in particular a graphics calculator as a medium of drawing and exploring.

Introduction

Edward Harrington Lockwood is known to posterity through his *A Book of Curves* (Cambridge University Press, 1961) which was an inspirational source of enrichment activities for students and teachers of the 1960s and 1970s. In it he collects and presents a range of standard curves, both familiar (parabola, ellipse, hyperbola) and strange (astroid, nephroid, right strophoid) and gives exercises for drawing them, often as the envelope of a family of straight lines or circles, and exploring their properties, as well as analytical summaries of their major equation forms and properties. Then in the second half he presents a number of standard ways in which families of new curves can be generated from others: conchoids and cissoids, roulettes, pedal curves, evolutes and involutes and many others.

The method of approach is the same in each section: an initial drawing, some consequential discussion via classical geometry, then further properties and drawing exercises, concluding with an analytical summary. In the Preface Lockwood explains his philosophy.

> Anyone who can draw a circle with a given centre and a given radius can draw a cardioid or a limaçon. Anyone who can use a set square can draw a parabola or a strophoid. Anyone who knows a few of the simpler properties of Euclid can deduce a number of properties of these beautiful and fascinating curves. [p. vii]

Thus we find the initial drawing leads in each case to some rather demanding deductive geometry. The geometry in fact is the key to the whole, with calculus and analytic geometry left unexplained as a resource for the more advanced student:

> The approach is by pure geometry, starting in each case with methods of drawing the curve. In this way an appreciation of the shape of the curve is acquired and a foundation laid for a simple geometrical treatment. There may be some readers who will go no further, and even these will have done more than pass their time pleasantly; but others will find it interesting to pursue the geometrical development at least to the point at which one or other of the equations of the curve is established. Those who have a

> knowledge of the calculus and coordinate geometry may prefer to leave the text at this point and find their own way, using as a guide the summary of results which will be found at the end of each chapter of Part I and some chapters of Part II. [loc cit]

For Lockwood and his intended readers, then, the path from initial exploration to full "understanding" was via classical geometry; analytical geometry and calculus were relatively unimportant.

School syllabuses and the students who study them have changed in many ways since Lockwood's time, and this last point emphasises one of them. The senior students of today are in general much more comfortable with an analytical argument or presentation than a geometric one. In Australia the discipline of geometry has changed considerably in emphasis, with much more observation and exploration and much less deductive argument than was the case in the traditional course Lockwood knew. Here is an example from the first chapter:

> SP' is joined. Then the points S, Q, q, P' are concyclic and angle AQS = angle qP'S.
> [p. 4]

This is in the middle of a paragraph of text wrapped around a pair of diagrams (which visually explain what is happening). To follow the actual argument in this last sentence the student has to recognize (1) triangle SQP' is right-angled at Q, and triangle SqP' is right-angled at q; (2) therefore both Q and q lie on the (semi)circle with SP' as diameter; (3) so SQqP' is a cyclic quadrilateral, and then (4) recognizing that angle AQS is an external angle of this quadrilateral and angle qP'S the opposite internal angle, the two angles are equal. Few students today would be able to appreciate the richness of implied argument in Lockwood's one brief sentence.

On the other hand the basic facts or analytical geometry and calculus are much more familiar to today's students than those Lockwood dealt with, and an analytical argument such as Lockwood presents among his "harder" material would today be the preferred means of understanding.

Technology has reinforced this change of emphasis. In Australia geometry is now more likely to have been encountered as an exploratory science through the wonders of Cabri or Geometer's Sketchpad, and almost every secondary student has become familiar with the basics of analytical geometry through use of a graphics calculator. These new tools can be used to render Lockwood's ideas more vivid: while some of the drawing exercises can still be stimulating and enjoyable, many are quite tedious or exacting and the use of technology has the same liberating effect as in statistics, where tedious calculations have been mostly eliminated through modern calculators and packages.

This paper is a brief exploration as to what might be possible in redeveloping Lockwood's material to suit the interests and abilities of today's students, using the technology familiar to them. I have taken the first chapter, on the parabola, and attempted to reconstruct some of the exercises and discussions to suit a student able to use a graphics calculator. Many other approaches are possible. One could duplicate most of the drawing exercises, sometimes quite spectacularly, in Cabri Geometry or Geometer's Sketchpad, but it would not then be possible to connect easily with analytical geometry and fit equations to the curves. At the other end a computer algebra system such as Mathematica or Maple could perform all of the constructions I have described with greater efficiency and in better graphics; but these are still not readily available to the bulk of our students at all times in the way the graphics calculator now is. Even given the choice of a graphics calculator, there are two different approaches to be taken. One could write the material for beginning students who can draw straight lines and circles, and use it as a powerful and stimulating introduction to new curves and to such "advanced" techniques as polar or parametric graphing; or

one could write more concise material for students already familiar with the possibilities of the calculator. For the sake of space in this paper I have taken the latter approach.

To Draw a Parabola
Lockwood introduces the parabola as the envelope of a family of straight lines:
> Draw a fixed line AY and mark a fixed point S. Place a set square UQV (right-angled at Q) with the vertex Q on AY and the side QU passing through S. Draw the line QV. When this has been done in a large number of positions, the parabola can be drawn freehand, touching each of the lines so drawn. [p. 3, with diagram]

This is still an easy and pleasant exercise for hand drawing, and can be easily reproduced in Cabri. On a graphics calculator we need a little preliminary analysis. For convenience we choose the x-axis to be the fixed line and the point (0, 1) to be the fixed point S. Place the vertex Q at the point (t, 0): then we need to draw the line through Q at right angles to SQ. The slope of SQ is $-\dfrac{1}{t}$, so the required line has slope t.

Thus we draw as many lines as required with equations $y = t(x - t)$. The parabola will then appear as the envelope [Fig 1].
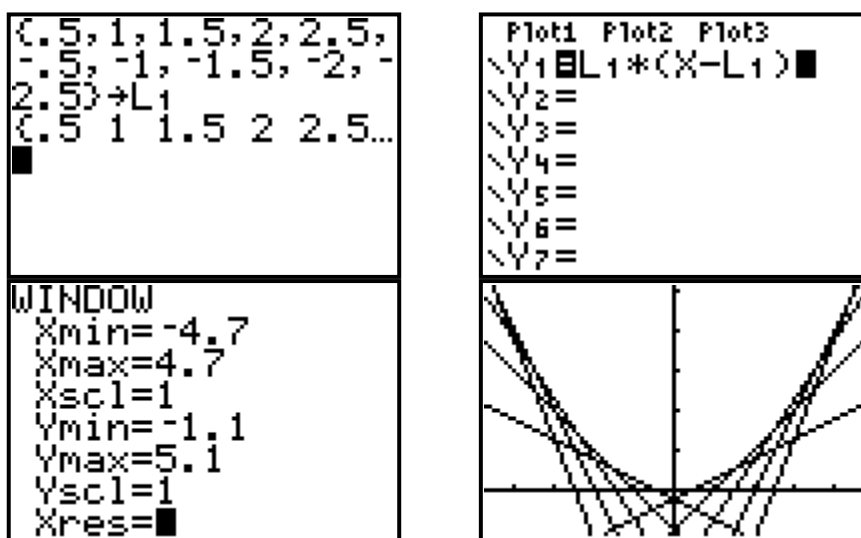

Fig. 1: Drawing the Parabola in familiar orientation

Students already familiar with the parabola in its basic forms might like to guess the equation of the parabola they can now see: it will be found to be $y = x^2/4$ (This type of guess-and-check is where using a graphics calculator is particularly helpful.)

Note 1: Lockwood follows the old convention in drawing the parabola "on its side" with a horizontal axis. For this view take the y-axis as the fixed line and the point (1, 0) as the fixed point S, and then draw several lines $y = x/t + t$. The resulting parabola is $4x = y^2$, or $y = \pm \sqrt{(4x)}$. [Fig. 2]
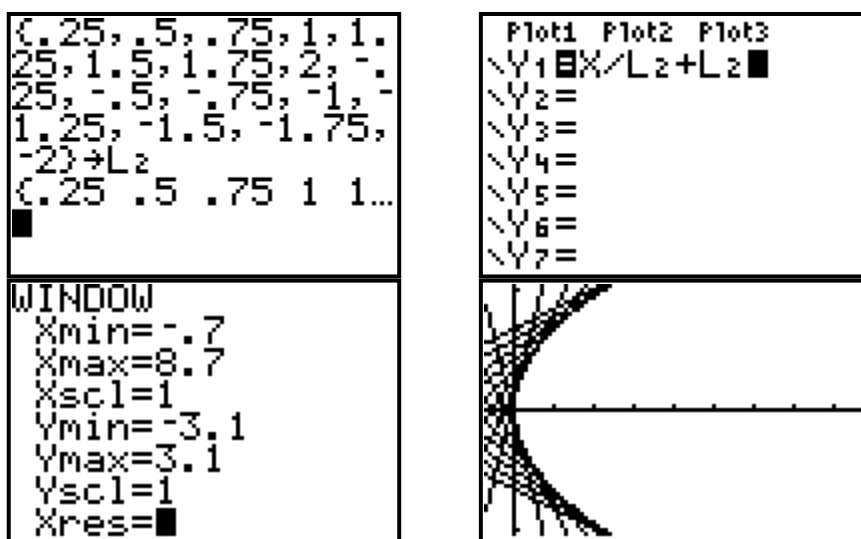
Fig. 2: Drawing the Parabola in traditional orientation

Note 2: The details of how the lines are to be drawn can be varied to fit the abilities of different students. More experienced students can use many other features of the calculator to draw families of lines, such as the Draw features, or writing a program; beginners could enter a number of different lines individually. In some cases these exercises could in fact be used to introduce some of the more advanced features such as the use of lists as shown.

Geometrical Properties

On the following pages (pp 4–5) Lockwood uses deductive geometry to develop the basic properties of the curve so produced, in particular deducing the focus-directrix property of the parabola. For today's students I would replace this section with an analytic argument to reach the same point, along the following lines:

The straight lines drawn do not themselves belong on the curve, but are each tangent to it at one point. Two of the lines will intersect at a point near to the final curve, and the closer the two lines are the closer their intersection point will be to a point on the curve.

The line that starts at x = s on the x-axis is

$$y = s(x - s)$$

and the line that starts at x = t on the x-axis is

$$y = t(x - t)$$

It is easy to see that the two lines intersect at

$$x = s + t, \ \ y = st$$

and if we now imagine s approaching t we see that the point actually on the curve must be at

$$x = 2t, \ \ y = t^2$$

so that the equation of the curve is indeed

$$y = x^2/4$$

Students who know calculus can then check that the tangent at the point x = 2t is as expected the line

$$y - t^2 = t(x - 2t)$$

or $\quad\quad\quad y = t(x - t)$

This establishes that the envelope of the lines is indeed the parabola.

To develop the classical properties of the parabola, we name the original fixed point S = (0, 1) as the *focus*, and choose an arbitrary point P on the parabola, say P = (2t, t^2): then the distance SP squared can be deduced to be

$$4t^2 + (t^2 - 1)^2 = (t^2 + 1)^2$$

so that SP = $t^2 + 1$, which is always one more than the y-value at P. If we then draw the line at y = −1 (the *directrix*) it is apparent that for each point P the distance from the focus S is equal to the distance from the directrix, the fundamental geometrical property of the parabola. Lockwood develops this property entirely from geometrical argument; for those with access to the text his Fig. 4, with four congruent triangles making a rhombus, is particularly illuminating. A dynamical presentation of the same idea would be an exciting prospect in Cabri or similar packages.

Cartesian and polar equations
From his detailed geometry Lockwood next deduces both the cartesian equation

$$y^2 = 4ax$$

and the polar equation

$$2a/r = 1 - \cos \theta$$

in each case for the parabola "on its side".

In my approach the cartesian equation is already known, and the polar equation (with the focus as pole and the line through the focus perpendicular to the directrix as initial line) can be easily deduced for those who know polar co-ordinates.

Further Properties
Lockwood next lists five additional facts about parabolas which "may be proved as exercises", in each case by classical geometry (with hints). The contemporary approach would be via analytical geometry and calculus, but each is still a useful and instructive exercise, although different sorts of guidance would be required. The first property is the reflecting property, that a line parallel to the axis cutting the parabola at P, and the line SP from P to the focus, make equal angles with the tangent at P (with instantly understandable applications in searchlights and telescopes, for instance). The second is that all normals to the curve cut the axis at a fixed distance from the foot of the perpendicular to the relevant point (see further below). The others deal with focal chords and tangents, parallel chords and so on.

All can be observed by using appropriate technology (a geometrical package or a graphical calculator, etc), but the "proofs" will require some deductive work.

Further Drawing Exercises
Next come nine further exercises labelled as "drawing exercises", although in some cases the point is more in deducing further properties of the geometry rather than the actual drawings. Several of these can be easily adapted to graphics calculators; others are rather more obscure or relate more obviously to hand drawing on paper.

As an illustration, I take the third of these exercises, which is to examine the evolute of the parabola, which Lockwood defines as the envelope of the normals to the curve. Further Property 2 gives some useful hints for hand drawing, but on the calculator there is no need for this: the normal to the curve at the point x = 2t, y = t^2 can easily be worked out to be the line

$$y - t^2 = -(x - 2t)/t$$

or
$$y = -x/t + t^2 + 2$$

and the envelope duly produced: we notice in passing that each normal cuts the axis (x = 0) at the point $y = t^2 + 2$, always 2 units further along the y-axis than the y-value of the point at which the normal is drawn (Further property 2). The envelope is shown [Fig. 3]
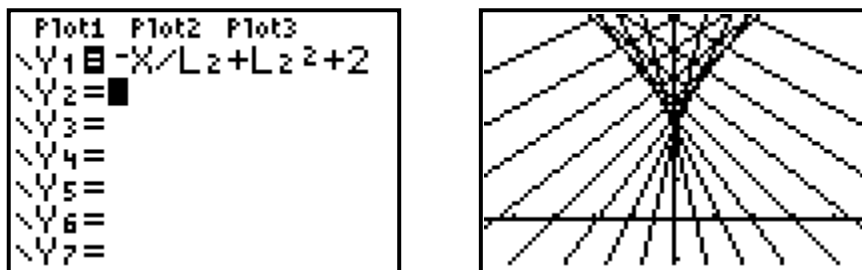

Fig. 3: Drawing the Semi-Cubical Parabola in familiar orientation

The curve produced is the *semi-cubical parabola*: proceeding as before we note that the normal at x = 2s meets the normal at x = 2t when

$$-x/s + s^2 + 2 = -x/t + t^2 + 2,$$

whence $\quad x = -st(s + t)$ and $y = st + s^2 + t^2 + 2$

so that the point on the envelope (when s and t coincide) is

$$x = -2t^3 \text{ and } y = 3t^2 + 2$$

So the equation of the envelope is $y = 3 (x/2)^{\wedge}(3/2) + 2$.

The instructive composite picture appears (in a different orientation) at the head of Lockwood's chapter: [Fig. 4]
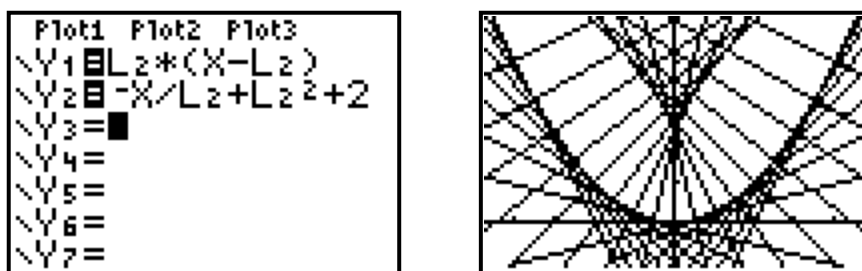

Fig. 4: Composite diagram

The Parabola: Summary

After the nine Further Drawing Exercises comes the Summary, an exhaustive list of 24 facts about parabolas in their cartesian, polar and parametric forms, and the elementary facts of their geometry. Most of these are not further elaborated; as explained in the Preface this section is intended for those "who have a knowledge of the calculus and coordinate geometry" and who "may prefer to leave the text at this point and find their own way." To spell out and derive each of these points would take several pages, even with modern technology; but they certainly contain a wealth of material. There then follows a page and more of historical material, and then a similar but shorter Summary of points on the semi-cubical parabola, with its own historical note to conclude.

In this short paper I have only been able to present a few examples of what might be possible, using just one of Lockwood's twenty-five chapters. Not every chapter has this lavish amount of material, but each has its own fascination. I am convinced that there is here (and in many other books of like

inspiration) a vast quantity of underutilised material which can be brought back to life for contemporary students with a little imagination.

One further paragraph from Lockwood's introduction demands to be quoted:

> Teachers may use the book in a variety of ways, but it has been written also for the individual reader. It is hoped that it will find a place in school libraries, and will be used too by sixth-form pupils, whether on the arts or the science side, who have time for some leisurely work off the line of their main studies, time perhaps to recapture some of the delight in mathematics for its own sake that nowadays so rarely survives the pressure of examination syllabuses and the demands of science and industry. [p. vii]

We should continue to entertain such hopes.

Reference:

Lockwood, E. H.: A Book of Curves (Cambridge University Press, 1961).