Integrating the Use of Computer Algebra into Traditional Mathematics Teaching

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Abstract.

This paper describes a research study that examines the impact of integrating the use of computer algebra with traditional mathematics teaching. The study has been conducted in a senior secondary school in New Delhi that follows the curriculum prescribed by the CBSE (one of the central boards responsible for education in India). In the prescribed curriculum the emphasis is on routine problem solving, developing 'by hand' skills and on gaining procedural knowledge rather than conceptual knowledge. Teaching with the help of technology is not a practice. This paper relates the experiences of the author in teaching with the help of computer algebra (specifically Mathematica) in such a traditional educational environment. The research focused on helping students visualize concepts, discover patterns and generalize results through Mathematica while developing 'by hand' skills thus maintaining a balance between the use of technology and paper-pencil techniques. The subjects of this study were 50 students of year 11 and 31 students of year 12. They were taught certain key concepts using Mathematica in a laboratory setting. This paper describes two such lab sessions. The first one deals with visualizing derivatives (year 12) and the second with visualizing quadratic functions (year 11). A structured feedback from the students at the end of each session revealed that Mathematica was instrumental in enabling them to obtain a deeper insight into the concepts. More importantly, the students' eagerness to attend similar lab sessions for other topics has provided encouragement for larger scale integration of CAS into traditional mathematics teaching.

Introduction.

In recent years much research has been directed towards exploring the relationship between computer algebra techniques and conceptual understanding in mathematics (Heid,1988; Palmiter,1991; Stacey,2001). Several issues have been studied related to the use of computer algebra systems (CAS) in mathematics instruction. Researchers are now faced with the following fundamental questions: How should CAS use be a part of the secondary school curriculum and how can it be used to improve conceptual understanding in mathematics? What is the role of paper-pencil skills in a computer algebra environment? What basic skills are necessary and in what amount? What kind of examination questions is relevant in a CAS environment? (Drijvers,1997,2000; O'Callagan,1998; Boehm, 2000; Kokol-Voljc, 2000; Kutzler, 1999; Heid,2001)

While CAS has been shown to facilitate the development of mathematical concepts, Heid (1988) suggests that its use may require re-sequencing of concepts and skills in mathematics courses. Visualization and exploration are key factors in concept formation in mathematics and the powerful graphing features of CAS can be suitably exploited for this purpose. However due to the use of CAS, development of 'by-hand' skills may be hindered. Hence research is being directed to find ways to integrate the use of CAS with paper-pencil techniques in teaching mathematics (Harwaardein and Gielen, 2001).

This paper reports my experiences in teaching with the help of computer algebra in a traditional educational setting where using technology for teaching mathematics is not a practice. At the school where I teach, however, innovation in teaching is encouraged. I am in-charge of a mathematics laboratory (an experimental project funded by the Department of Education) and one of its objectives is to explore the use of technology for teaching mathematics. The laboratory is equipped with computers, CAS (Mathematica) and graphing calculators (HP38G). It has been my attempt to augment my regular classroom lessons with sessions in the mathematics laboratory where Mathematica is used as a vehicle for exploration, visualization and demonstration. The emphasis is on developing concepts rather than on developing manipulative skills. This paper describes two lab sessions. The first one was conducted with students of year 12 where they were introduced to the concept of the derivative and were made to explore some rules for derivatives. In the second lab session students of year 11 explored quadratic functions and equations.

Lab Session 1: Visualizing the Derivative.

Educational Setting and Background Knowledge of Students.
In the prescribed curriculum calculus forms a part of the syllabus at the senior secondary level. Functions are taught in year 11 and limits, continuity, differentiation and integration are taught in year 12. However, the focus is on routine problem solving techniques. Examinations and assignments focus on testing these techniques. The emphasis is therefore on developing procedural knowledge and 'by-hand' skills rather than conceptual knowledge.

31 students of year 12 participated in this lab session. In the classroom I taught in the traditional manner (like all my other colleagues) using only chalk and board and in the lab I taught using Mathematica. Before going through this lab session the students had acquired the following concepts and skills:
(i) Concept of function, domain and range of a function and operations on functions.
(ii) Concept of limit of a function and evaluation of limits (by hand).
(iii) Concept of continuity of a function.
Differentiation was the next topic in the sequence. I decided to teach the concept of the derivative through Mathematica in the lab. At the beginning, the students were made to go through a one-hour session in which they were familiarized with some of the important features of the package such as plotting graphs, solving equations etc. They were made familiar with commands such as **`Plot`**, **`Solve`**, **`Limit`** and **`Table`**. A handout giving the syntax for these commands was provided to each student. During the lab sessions two to three students had to share one machine.

Aim of Lab Session.
The aim of the lab session may be summarized as follows:

a. To introduce the student to the concept of the derivative.

b. To highlight the difference between the notions of 'average rate of change' and 'instantaneous rate of change'.
c. To enable the student to compute derivatives using first principles.
d. To visualize the derivative function by plotting graphs and to find the rules for the derivatives of some elementary functions.

Method and Teaching Sequence.

I used a step-by-step sequence to introduce the concept of the derivative. Chalk and board were used for explanations and students were asked to use Mathematica when visualization was required.

(a) The first step in the sequence was to explain the physical interpretation of the derivative for which I used the familiar method demonstrated in Figure 1.

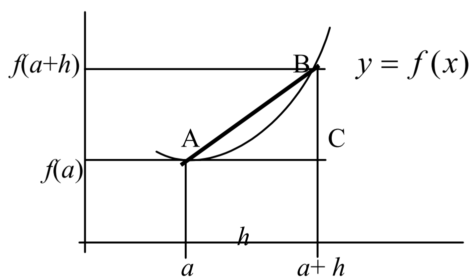

Figure 1: A chalk and board diagram for explaining the concept of the derivative.

The students were demonstrated how the slope of the secant, $\frac{f(a+h)-f(a)}{h}$, changed as point B approached A. Thus in the limiting process as $h \to 0$, the secant AB becomes a tangent to the graph at A. We say that the derivative of the function $y = f(x)$ at $x = a$ is the slope of the tangent at A, and is denoted by $f'(a)$. The definition of the derivative at $x = a$ was then introduced formally as $f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$. In general the derivative of $y = f(x)$ was introduced as $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$ (provided the limit exists).

(b) I emphasized the difference between the average and instantaneous rate of change using Figure 1 above. The students were made to understand that the slope of the secant line gives the *average change* of the function $y = f(x)$ over an interval while the slope of the tangent line gives the *instantaneous rate of change at a point*.

(c) In this step I focused on developing the 'by-hand' skill of finding the derivatives of simple functions using *first principles*, namely the definition of the derivative. As an illustration I computed the derivative of $f(x) = x^2$ on the board as follows.

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h\to 0} \frac{2xh + h^2}{h} = \lim_{h\to 0} 2x + h = 2x.$$

(d) In order to visualize the derivative function for $f(x) = x^2$, I asked the students to resort to Mathematica. The following code was used to plot the expression $\frac{(x+h)^2 - x^2}{h}$ for various values of h, approaching 0, on the same set of axes. The plot of $2x$ is also shown on the same axes in red. (I wrote the code on the board and explained the significance of each line.)

```
Clear[f,x]
f[x_]=x^2;
plot1=Plot[{(f[x+5]-f[x])/5,(f[x+3]-f[x])/3,
(f[x+1]-f[x])/1,(f[x+0.8]-f[x])/0.8,(f[x+0.4]-f[x])/0.4,
(f[x+0.1]-f[x])/0.1},{x,0,10},PlotStyle->Thickness[0.001]]
plot2=Plot[2*x,{x,0,10},PlotStyle->RGBColor[1,0,0]]
Show[plot1,plot2]
```

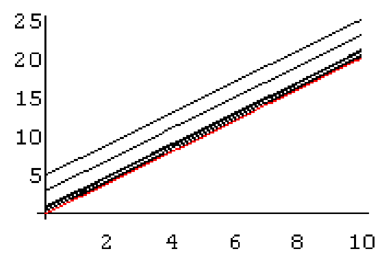

Figure 2: The Mathematica output indicating that the plots of $\frac{(x+h)^2 - x^2}{h}$ for h equal to 5, 3, 1, 0.8, 0.4, and 0.1 (shown in black) approach the graph of $2x$ (in red), as h gets smaller. Each line is for a distinct value of h).

Observing the output the students concluded that as h gets smaller, $\frac{(x+h)^2 - x^2}{h}$ approaches the derivative function $2x$.

In order to explore the derivative at a specific point ($x = 0$), the students were made to draw four secants on the graph of $f(x) = x^2$. These secants joined the origin to the points (2,4), (1.5,2.25), (1,1) and (0.5,0.25) respectively. The students observed that their slopes 2, 1.5, 1 and 0.5 approached 0 which was the slope of the x-axis (the tangent to $f(x) = x^2$ at $x = 0$). This observation led the students to conclude that the slopes of secant lines came closer and closer to the slope of the tangent at $x = 0$ as h got smaller (see Figure 3 for the Mathematica output). Thus the derivative of $f(x) = x^2$ at $x = 0$ was 0 (the slope of tangent at $x = 0$) which could also be obtained by substituting $x = 0$ in the formula for the derivative function $2x$. This exercise helped to highlight the relationship between derivative at a point and the general derivative function. In future lab sessions on visualizing derivatives, it may be worthwhile to explore of the derivative of a function at a point before exploring the derivative function.

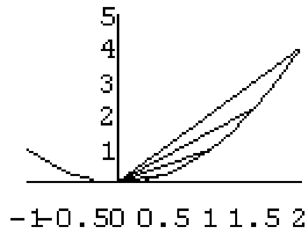

Figure 3: Mathematica plot showing that the secant lines (joining the origin to various points on the parabola) approach the tangent to the graph of $f(x) = x^2$ at $x = 0$ (the x-axis).

(e) For further exploration the students were asked to explore the derivatives of $f(x) = x^3$ and $f(x) = x^4$ by making minor modifications in the above program. They were instructed to replace **`2*x`** by **`f'[x]`** since they did not know the rules for the derivatives of these functions. On seeing the outputs some responses were:

The derivative of $f(x) = x^3$ must be some ax^2 because the graph is shaped like a parabola just like $f(x) = x^2$ but I can't guess the value of a.

For $f(x) = x^4$ the derivative looks like a cubic function but I can't guess what it is exactly.

At this point they are asked to compute the derivatives of $f(x) = x^3$ and $f(x) = x^4$**,** by hand, using first principles. To verify their answers they used the **`D`** command (**`D[x^3,x]`**and **`D[x^4,x]`**) **.**The aim of this exploration was to lead the students to find a formula for the derivative of $f(x) = x^n$**.** Mathematica's **`Table`** command was used to evaluate the expression $\lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$ for n varying from 1 to 6. The output enabled the students to generalize the result that the derivative of x^n is nx^{n-1}.

```
Table[{x^n,Limit[((x+h)^n-x^n)/h,h->0]},{n,1,6}]
```

$\{\{x,1\},\{x^2,2x\},\{x^3,3x^2\},\{x^4,4x^3\},\{x^5,5x^4\},\{x^6,6x^5\}\}$

In a subsequent lab session the students were made to understand that continuity at a point did not necessarily imply differentiability. They were made to plot the function $f(x) = \max\{(x-2)^2, x\}$ for x ranging between 0 and 5. Visual observation showed them that the function was continuous within the given range but there were two points (x = 1 and x = 4) where $f(x)$ did not have a unique slope that is the derivative did not exist. This helped to introduce the concept of left hand and right hand derivatives and the fact that when these are unequal the derivative does not exist at that point. The two points (x = 1 and x = 4) were determined by solving the equation $(x-2)^2 - x = 0$. The students used first principles to find the left and right hand derivatives at these points, and concluded that these were different. They also concluded that when the graph had a 'corner' the derivative did not exist at that point (see Figure 4 for Mathematica output).

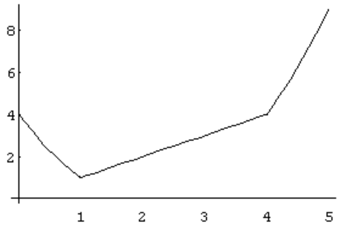

Figure 4: The Mathematica plot of $f(x) = \max\{(x-2)^2, x\}$.The derivative does not exist at the 'corners' ($x = 1$ and $x = 4$).

Results.
At the end of the lab sessions which consisted of three 35 minute periods, a short test of 30 marks was administered in which all problems had to be solved by hand without using Mathematica. The first question required the students to draw the graphs of some functions (such as $2x^3$ and $\sin 2x$) and their derivatives while the second question required them to compute the derivatives of functions such as $cos3x$, e^{x^2} and $\sin\sqrt{x}$ using first principles. I administered the same test on another group of 32 students who had been taught derivatives in the traditional manner without Mathematica. While the scores of both groups were comparable for the second question, the Mathematica class performed significantly better than the traditional class in the first question. In the traditional class almost 50% of the students did not attempt question 1. The Mathematica class had a mean score of 22.7 whereas the traditional class had a mean score of 17.5.

Students' Response.
At the end of the lab sessions a written feedback was taken from the students. Some comments among various responses were:
'*Mathematica has given me a visual feel for the derivatives of various functions. Although (without using Mathematica) I could find the derivatives by first principles, these were only symbols to me.'*
'Mathematica helped me to actually see how $\frac{f(x+h)-f(x)}{h}$ *approaches the derivative function for smaller and smaller values of* h'.
To conclude it may be appropriate to say that this lab session gave the students a visual feel for the derivatives of some important functions and enabled them to 'discover' the rules for these derivatives on their own. Here Mathematica was used more as an investigative tool for exploration and while the emphasis was on visualizing the derivatives of various functions, there was no compromise on finding the rules for the derivatives using paper-pencil techniques (first principles). Thus a balance was achieved between the use of Computer Algebra and the development of 'by-hand' skills.

Lab Session 2: Visualizing Quadratics.

Educational Setting and Background Knowledge of Students.
This lab session was conducted with 50 students of year 11. The students had prior knowledge of the following:

(a) the concept of a function, its domain and range, and types of functions (one-one, many-one, onto and into).
(b) linear and quadratic functions.
(c) quadratic equations, formula for finding the roots of a quadratic equation, the nature of roots and their relationship with the discriminant $b^2 - 4ac$.
Before attending the lab all 50 students underwent a one-hour session on Mathematica where they were familiarized with plotting graphs of functions, solving equations and fitting curves to data (using the `Plot`, `Solve` and `Fit` commands respectively.)

Aim of Lab Session.
The aim of the lab session may be summarized as follows:
(a) To enable the student to visualize quadratic functions.
(b) To fit a quadratic function to a given data.
(c) To understand the significance of the coefficients of a quadratic function.
(d) To visualize the nature of roots of a quadratic equation.

Method and Teaching Sequence.
Each student was given a worksheet comprising of three activities.
Activity 1
In the first activity the following problem was posed.
A temperature probe is placed in hot water. Initially the temperature is recorded at 98^o C. The temperature is then observed for the next 4 minutes in intervals of 30 seconds and the readings are tabulated as below. Here t denotes the time in minutes and $y(t)$ denotes the temperature at time t in degrees Centigrade.

t	**0**	**0.5**	**1**	**1.5**	**2**	**2.5**	**3**	**3.5**	**4**
$y(t)$	**98**	**93.25**	**89**	**85.25**	**82**	**79.25**	**77**	**75.25**	**74**

The first exercise required the students to find the quadratic function $y(t) = at^2 + bt + c$ which fits the data. They were instructed to do this by hand (by choosing any three points and substituting them in $y(t) = at^2 + bt + c$) and then verify their answer using Mathematica. The students came up with the equations $c = 98$, $a + b + c = 89$, $4a + 2b + c = 82$ by choosing the points (0,98), (1,89) and (2,82). On solving they obtained the values as a =1, b = -10 and c = 98 and concluded that $y(t) = t^2 - 10t + 98$ was the required quadratic function. In the second exercise they were asked to experiment with different sets of three points from the data. Each time they obtained the same equation and finally verified their answer using the `Fit` command. `ListPlot` and `Plot` commands were then used to plot the data and the quadratic function. This output of the program (Figure 5) convinced the students that the quadratic function obtained by them actually passed through the given data.

```
data={{0,98},{0.5,93.25},{1,89},{1.5,85.25},{2,82},
{2.5,79.25},{3,77},{3.5,75.25},{4,74}};
quadfit=N[Fit[data,{1,x,x^2},x],2]
```

```
plot1=ListPlot[data,PlotStyle->PointSize[0.07]]
plot2=Plot[quadfit,{x,0,5}]
Show[plot1,plot2]
```

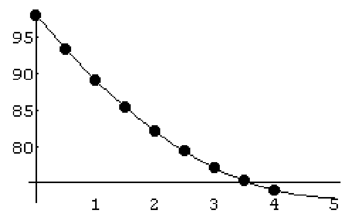

Figure 5: The Mathematica output indicating that the quadratic fit $y(t) = t^2 - 10t + 98$ passes through the data points.

Activity 2

In the next activity the students were made to explore the quadratic function $y = a(x-h)^2 + k$ by varying the constants *a*, *h* and *k*. In the first exercise they explored the effect of varying *a* by plotting the graphs of $y = ax^2$ for various values of *a* and comparing these with $y = x^2$. The Mathematica outputs led them to conclude that the parabola narrowed for $a > 1$, widened for $0< a <1$, opened upwards for $a >0$ and opened downwards for $a < 0$. In the second exercise, the students observed the effect of varying *h* by plotting $y = (x-h)^2$ for various values of *h*. By comparing the plots with $y = x^2$ they concluded that varying *h* caused the parabola to shift by *h* units on the positive the *x* – axis (for $h>0$) and by *h* units on the negative *x*-axis (for $h<0$). In the third exercise the students plotted $y = x^2 + k$ for various values of *k*. They concluded that the parabola shifted upward on the positive *y* – axis by *k* units for $y = x^2 + k$ and shifted downward on the negative *y* – axis by *k* units for $y = x^2 - k$. In a subsequent lab, students explored the effect of varying *k* in the function $y = k - x^2$. This led to a sequence of downward opening parabolas with vertices on the *y*-axis (see first plot in Figure 6). Then they plotted $y = \left|k - x^2\right|$ separately for different values of *k*. They observed that the graphs were above the *x* –axis and were combinations of the positive portions of the graphs of $y = x^2 - k$ and $y = k - x^2$ (see second, third and fourth plots in Figure 6).

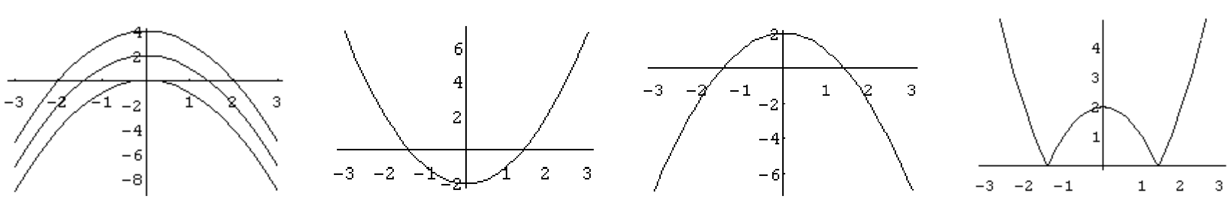

Figure 6: Mathematica plots of the functions $y = k - x^2$ (for k =0,2,4), $y = x^2 - 2$, $y = 2 - x^2$ and $y = \left|2 - x^2\right|$ (in that order).

Activity 3

The aim of this activity was to enable the students to see the relationship between the nature of roots of a quadratic equation and its corresponding quadratic function. They were asked to state the nature of roots of the quadratic equations $x^2+4x+4=0$, $x^2+x-6=0$, $x^2-4x+13=0$ using the discriminant $\Delta=b^2-4ac$ (this had been taught in the regular classes). After identifying the nature of roots, they calculated the roots and plotted the three quadratic functions corresponding to the above equations. The first parabola touched the x – axis at (-2,0). The second parabola cut the x – axis at the distinct points (-3,0) and (2,0), while the third parabola did not cut the x – axis at all. At the end the students concluded that if a quadratic equation.

- has real and equal roots the corresponding parabola touches the x – axis at a single point (or two coincident points).
- has real and distinct roots the parabola cuts the x – axis at two distinct points.
- has imaginary roots the parabola does not cut the x – axis at all (cuts it at imaginary points).

In a subsequent lab, the students were made to explore the effect of varying k in the function $y=ax^2+bx+k$. This led to a sequence of parabolas with vertices on the same vertical axis.

Students' Response.

This lab session took three full 35-minute classes at the end of which a written feedback was taken. The most encouraging comment given by a group of four students is as follows:

'*These lab sessions with Mathematica have truly reinforced our concepts, which in our view, holds greater importance than performing routine calculations. The very fact that a dry subject like quadratic equations can be made lively and realistic through graphs and explorations, has immensely broadened our view of mathematics.'*

To summarize it would be appropriate to say that this lab provided the students with a visual feel for quadratic functions. Plotting graphs by varying the constants enabled the students to make observations on their own. Since all the three activities required students to perform some calculations on paper, the development of 'by-hand' skills was ensured.

Concluding Discussion.

In this paper I have tried to describe my experience of using CAS (Mathematica) for teaching mathematics to senior secondary students in a laboratory environment. The purpose of using Mathematica was to enhance students' exploration skills by allowing them to visualize and experiment. Integrating Mathematica with traditional teaching methods in the form of lab sessions served the following purposes:

(a) *Visualization*: Mathematica helped to illustrate a concept, a fact or a process that would be difficult to explain using only chalk and board. It would, for example, be difficult to illustrate how the expression $\dfrac{f(x+h)-f(x)}{h}$ approaches the derivative function for smaller and smaller values of h without Mathematica.

(b) *Exploration*: Mathematica enabled the students to explore concepts, discover patterns and generalize results on their own or with some guidance from me. It gave them a sense of discovery, which would be difficult if not impossible to achieve in the traditional classroom.

(c) *Simplifying the teaching –learning process*: The use of Mathematica required me to re-sequence the concepts and ideas thus breaking down the learning process into smaller, easier digestible pieces.
(d) *Sustaining students' interest*: A general observation was that the student's interest could be sustained for a longer period of time by integrating Mathematica into the lesson. In the lab it was easier to get them to do the 'drill' by hand than in a traditional 35-minute class.

To conclude, integrating the use of CAS with 'by-hand' skills in a traditional teaching environment led to a deeper conceptual understanding among students and greater enthusiasm was generated among them towards learning the subject matter. My colleagues at school expressed a more positive attitude towards the use of technology in teaching mathematics and this left me with a great sense of achievement. This experiment has paved the way for integrating CAS with regular teaching on a larger scale.

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