The Strength of Graphical Elements in Pre-Calculus

Nur Azman Abu and Faqir M Bhatti
FIST, Multimedia University, Melaka 75450, Malaysia.
E-mails : azman@mmu.edu.my; faqir.bhatti@mmu.edu.my

Abstract
Over the past more than two decades, the population of students entering universities has significantly increased to the point where mass lectures have to be delivered to hundreds of students. More emphasis has been focused on teaching technical sciences and engineering in these days. The main pre-requisite for all these scientific disciplines is Calculus. Many techniques have been developed at high school as well as tertiary level to teach calculus around the globe using various software with personal computer, because of low cost of quality software and hardware. One of the popular techniques, which are being used in teaching Calculus, is the Graphical Techniques. We argue that the technique can be used in the latest concept of 'Smart School', to teach Calculus and particularly the 'Additional Mathematics'. We put forward our argument that graphical techniques can change the traditional activities in teaching and learning of Additional Mathematics and particularly Calculus, trigonometry and geometry, because of visual concepts. This could lead to a smooth transition from manual to computer aided learning approaches.
Key Words: Graphical Techniques, Smart School, Visual concept.

Introduction
Much attention has been paid in the recent years to show the natural introduction of Pre-calculus into Additional Mathematics via its graphical representation and visualization. Mathematical visualization(see Bhatti et al, 2001 and for more reference) is the art of creating a tangible experience with abstract mathematical objects and concepts. The advent of high-performance interactive computer graphics systems has opened a new era, its ultimate significance can only be imagined(Abu et al., 2001).

The chief aim of this paper is to focus on the most part an IT(Information Technology) perspective on the progress, techniques, and prospects of mathematical visualization, emphasizing the concept where interactive paradigms are of growing importance(Hanson et al., 1994).

First, we give briefly the general background and then focus on some of the visualization techniques that have been used to model mathematical problems. Examples of images are supplied throughout this paper to clear the concepts. The graphs illustrating ideas are not only pretty but also make the concept much more basic and easier to understand. It is hoped that these graphs will be, not only aesthetically appreciated, but also fundamentally understood. They are comparable to graphs of functions of a real variable, which are understood by most people who would have learned any course on Calculus during their early education or late at university level.

Due to tremendous paradigm shift that technology has brought in the recent years, instruction in Mathematics will have to ‘catch up’ with the new era or otherwise be increasingly irrelevant. Mathematics classes should focus more on cooperative learning, problems solving, and investigative learning mode as an important part of education. Simultaneously with the aid of computer visualization, the horizon of teaching Mathematics is consistently broaden. Nevertheless it still requires the art of drawing pretty and nice functions to be presented.

Pretty Visualization

Since a long time ago, symmetric geometry was a popular pattern design. The Fractals(See Lanius 1999, and Sprott) have become the object of interest because they look very pretty when generated by computer graphics for the last more than twenty years.. Teachers can use fractal geometry to introduce students to reinforce the arithmetic and geometry skills that they study in school. The lessons are designed for students to work independently or with guidance from the teacher. The benefits of the lesson is to be able to integrate of Mathematics and art of making use of arithmetic and geometry through visualization.

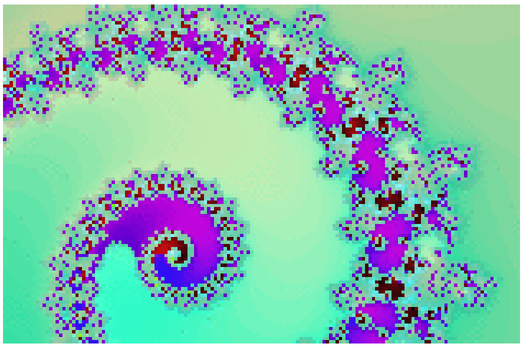

Figure 1. A sample of fractal picture.

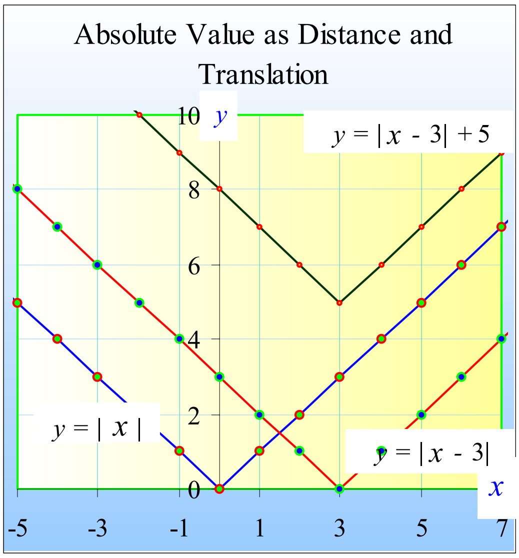

Figure 2*a.* The translation of a line.

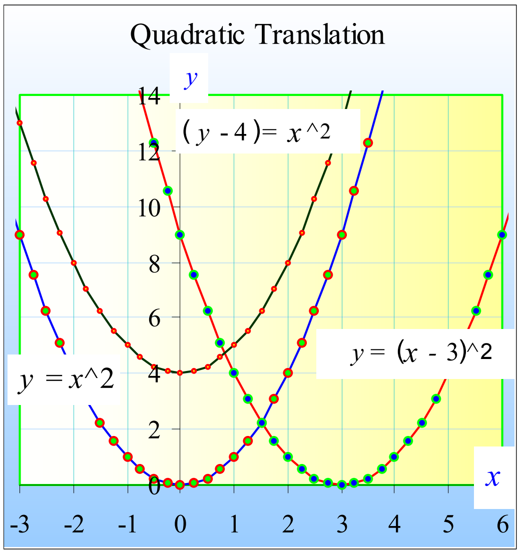

Figure 2*b.* The translation of quadratic function.

There is serious Mathematics behind the pretty pictures, and moreover, much of it is quite accessible to secondary school students. Furthermore, the Mathematics behind the images is often even prettier than the pictures themselves as shown in *Figure* 1 (see Lanius, 1999).

Information Technology can be potent mathematical tools that engage learners in meta-cognitive reflection. Proponents of computer-aided mathematical visualization argue that visualization can help building the intuition necessary to understand mathematical concept. Computer graphics is a useful tool for teaching simple topics in Mathematics(Abu et al., 1998).

Computer Algebraic System

In the mid-eighties the availability of Computer Algebraic System(CAS) for personal computers attracted mathematics educators to the possibility of using them in the classroom. CAS technology, with its powerful combination of numeric and symbolic computation, colourful graphics, and programming facility, is a natural and logical continuation of the scientific and graphical calculators.

The recent approach of CAS has inspired many educators of Calculus all over the world(Belmonte & Yasskin, 1999). It has, however, brought complications of its own. It requires more reading and insight from the student than traditional Calculus. This is the very reason why an average student experiences difficulties with this new approach.

More guidance should be given to assist the student in improving his reading skills in Mathematics and to cultivate the required insight. For a mathematics educator nowadays, it is important to be able to explore mathematics via CAS.

In programming, the common question to ask is "What is your first programming language?"
In Mathematics, it becomes
"Which CAS do you use?"

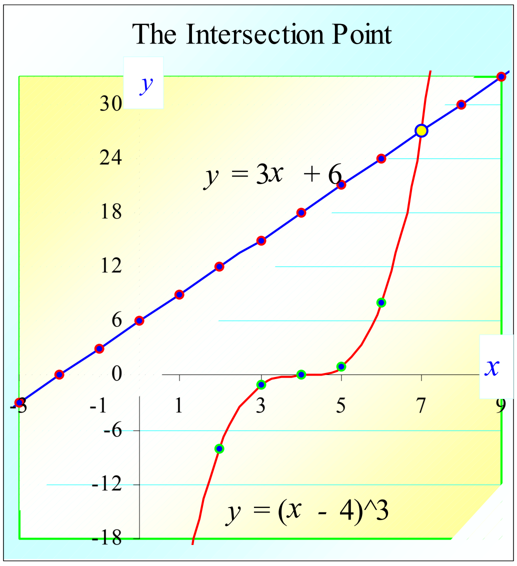

Figure 2c. The graph of the linear line and cubic curve can be sketched before computing the intersection point.

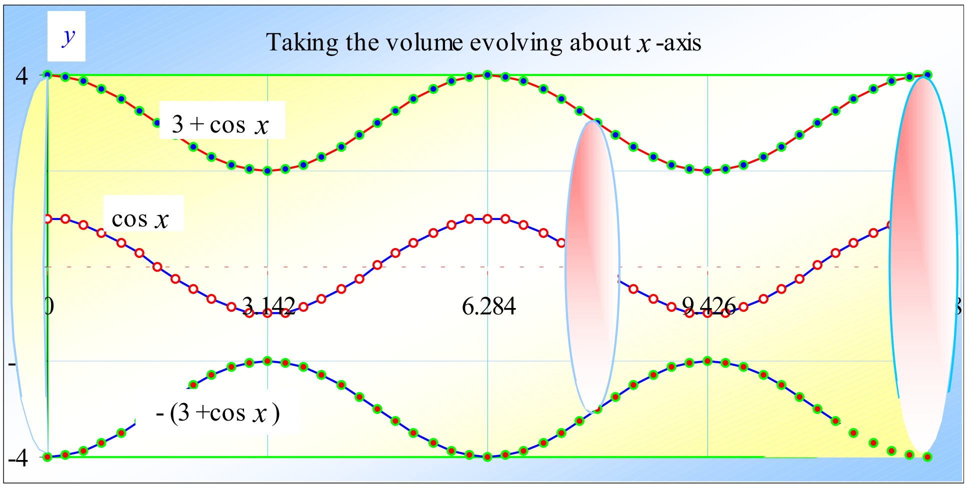

Figure 3: The cosine function has been moved 3 units upward and reflected with respect to x-axis. The region under the graph can then be rotated about the x-axis.

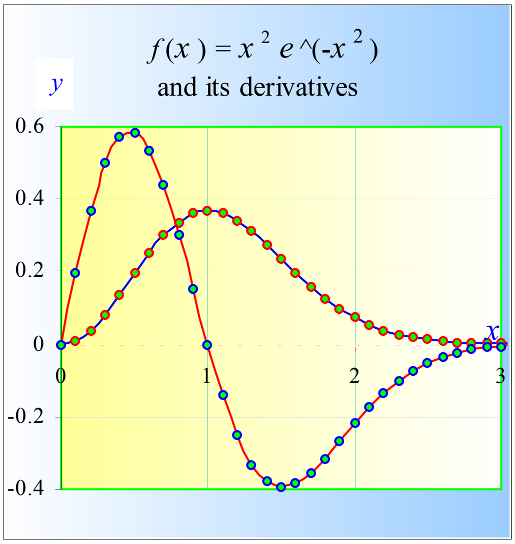

Figure 4*a*. Sketching the graphs of $f(x) = x^2 \cdot e^{-x^2}$ and its derivative.

There are several other topics already available in Modern Mathematics which can be extended to solve two simultaneous equations. In *Figure* 2*c*, we give the concept of intersection point of y = 3*x* + 6 and y = $(x-4)^3$. This graphical concept should be promoted besides the formulation techniques at this early stage.

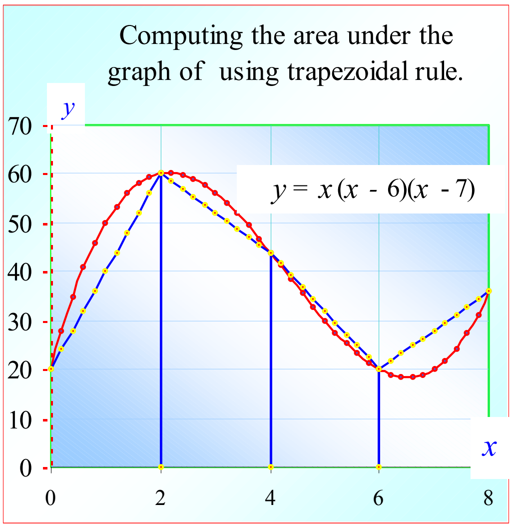

Figure 5*a*. Approximating the area under the graph using Trapezoidal rule.

Modern Mathematics

Modern Mathematics has been known to be much easier than Additional Mathematics. This is partly due to its simple mode of delivery and intermediate representation on what is going on in classroom. While the topics of translation, reflection and rotation are common in Modern Mathematics(Huraian Sukatan, 1998), the same concepts can be applied in a casual manner as well in Additional Mathematics. In *Figure* 2*a* and 2b, we try to show the translation of linear and quadratic functions.

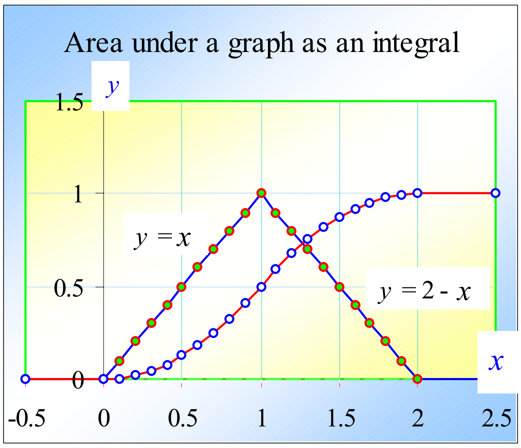

Figure 4*b*. The integral of a simple piecewise function

$$f(x) = \begin{cases} x, & 0 < x \le 1 \\ 2 - x, & 1 < x < 2 \\ 0, & otherwise \end{cases}$$

In Malaysia, the de-emphasis of trigonometric functions from the Additional Mathematics in the last one decade has caused significant difficulties for the students to cope with the Calculus syllabus at the university level. The trigonometric and transcendental functions must be reintroduced. And graphical sketching will help. In *Figure* 3, we give an example of cosine function moves 3 unit upward and evolves about *x*-axis.

Learning Activities

Physics and Chemistry subjects use experiments in their learning activities. Additional Mathematics should do the same with the help of Information

Technology. The days are gone when teachers just used to solve problems in the class without any practical in Mathematics. In fact, using any software like Microsoft Excel in the class room will give students the practical concept in Mathematics The selected topics such as derivatives and integrals will be much more meaningful in the activities to investigate the underlying concept before introducing the limit at the pre-university level.

In the mean time, students may be asked to compute and sketch the derivative of a function manually. A sample in *Figure* 4*a* shows that for each interval they can compute the slope between the two points for each interval and sketch the resulting graph by connecting the points. This kind of activity is preferably done in groups. The students should be encouraged to discuss on what they are computing. Then, students have the impression on how a function it looks like. They will get clearer picture of the function and its visual properties. It will be a good practice for the students to use any suitable graphical calculator to compute a given function. The students may be given a function and asked fill up a plot table.

Integrals can also be illustrated in a much more livelier mode, as visually mapped out in *Figure* 4*b*. It may not appear to be practical in manual activities until the image is enlarged ten folds.

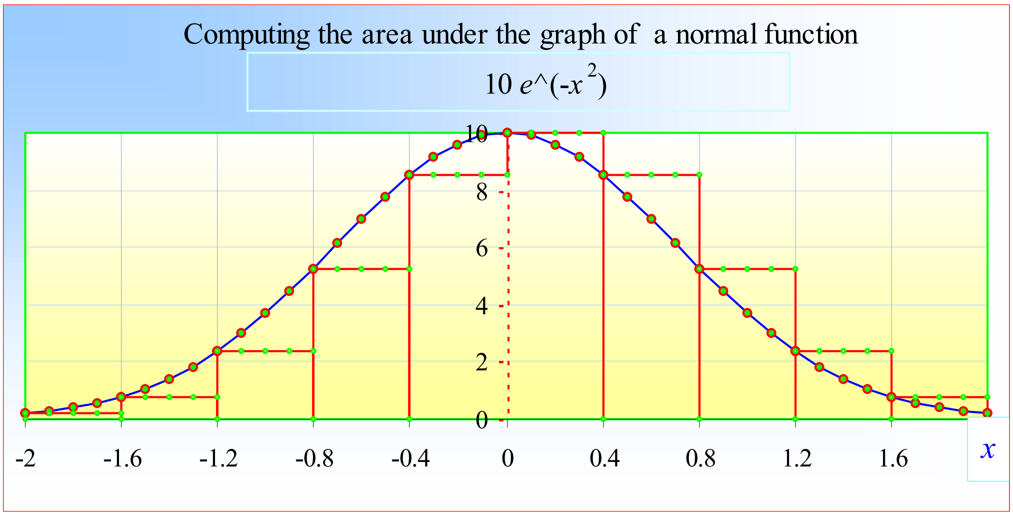

Figure 5*b*. Area under a graph of a normal function $10e^{-x^2}$ via Riemann Sum.

Learning activities should incorporate multiple representations of mathematical topics and/or multiple approaches to representing and solving mathematics problems. Furthermore, use of technology allows students to set up and solve problems in diverse ways, involving different mathematical concepts, by removing both computational and time constraints(Garofalo et al., 1998).

An early introduction of numerical integration can both consolidate the conceptual understanding of the definite integral and directs the students' attention immediately from Trapezoidal area under the graph as in *Figure* 5*a* to Riemann sums as in *Figure* 5*b*.

Take advantage of technology to enhance teaching

Learning activities should not only take full advantage of the capabilities of technology but also extend beyond what can be done without technology. Using technology to teach the same mathematical topics, in fundamentally the same ways, that could be taught without technology, does not enhance students' learning of mathematics. It just relies on the usefulness of technology(Abu et al., 1999). Furthermore, using technology to perform tasks that are just as easy or even better carried out without technology may actually be a hindrance to the learning process.

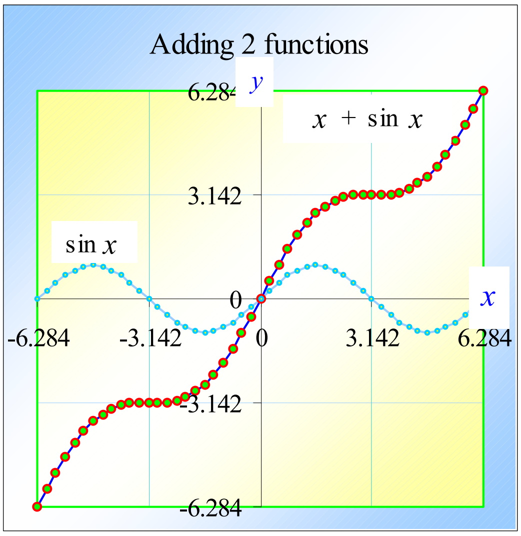

Figure 6*a*. Visually adding the Sine and a simple function $f(x) = x$.

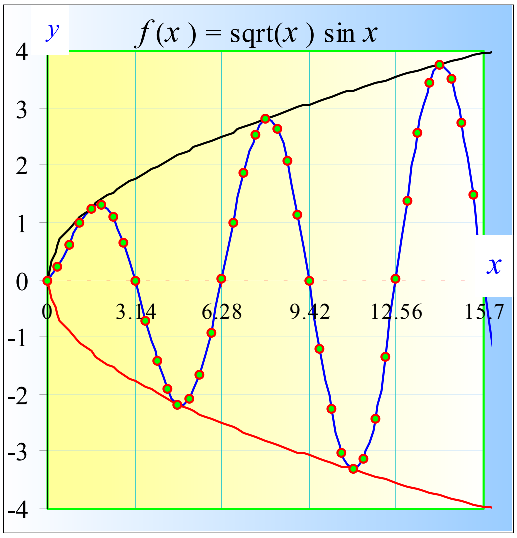

Figure 6*b*. The sine function within the cave of simple functions $\sqrt{x}$

Algebraic Functions

Function is rapidly becoming one of the organizing concepts within mathematics education (Fukuda et al., 2000). The days are long past when just numerical computation is adequate. Students should not only apply a particular function but they are also encouraged to create new functions which adequately represent the problem. In *Figure* 6*a*, it shows the graphs of basic sine function is being added along the line $y = x$.

Basic algebraic operations employ two powerful representation of functions symbolically and graphically. The art of visual technique provides both symbolic and graphical representations of functions. We give below 5 basic algebraic operations for two single variable functions f and g(Abu et al, 2001),

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(f*g)(x) = f(x)g(x)$$
$$(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$$
$$(f \circ g)(x) = f(g(x))$$

The graphing technology offers an outstanding opportunity for teachers to present the function concept in a meaningful manner. It is common to have differentiation problem using product rule in Calculus. Having seen the shape of the function themselves will add an intuition in what they are really working on. We give a typical example *Figure* 6*b*, which shows the shape of the function $\sqrt{x}\,\sin x$. The product gives a shape which is hard to imagine without drawing it.

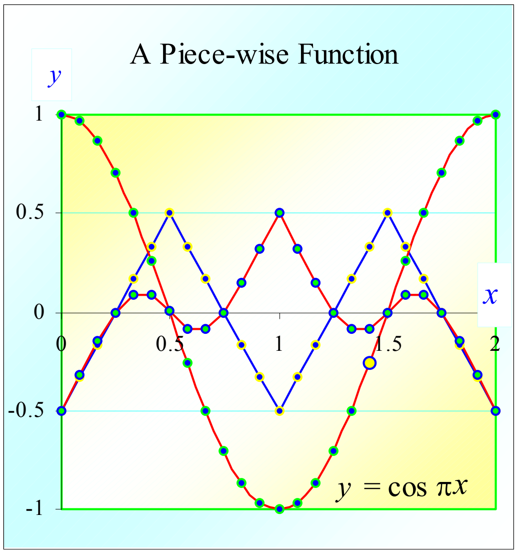

Figure 7. A simple algebraic function $(f * g)(x)$ where the f is A zigzag and g is a Cosine function.

Piecewise Functions

The notion of piecewise function is uniquely suited for teaching and learning with the above techniques. With an understanding of the various families of functions, coupled with real world phenomena, connections can be established. Students shall begin to see mathematical models as pictorial representations that have meaning.

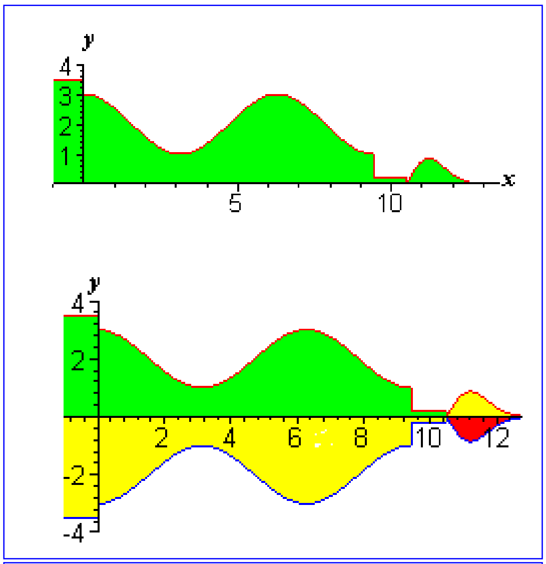

Figure 8*a*. Another sample of piecewise function

$$f(x)=\begin{cases} 3.5, & x<0 \\ 2+\cos x, & 0<x<3\pi \\ 0.2, & 0<x<10.5 \\ 2(x-10.5)e^{-(x-10.5)^2}, & x>10.5 \end{cases}$$

and its reflection upon the *x*-axis.

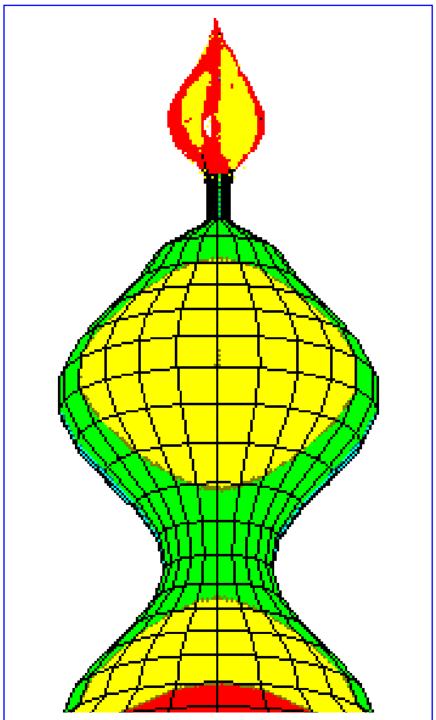

Figure 8*b*. The function f is rotated around the ***x***-axis before being flipped upright to form a traditional lamp.

The product of two functions where one of which is piecewise can be illustrated as in *Figure* 7. These are just few examples of how to enhance students understanding of concepts by giving them visual representation to accompany the verbal presentation. The students should be guided to explore the properties of algebraic operations on their own. This may aid deeper understanding and longer retention of ideas in their mindset.

It is possible nowadays to show interesting *new broader views*, thereby extending the teaching of mathematics such as in combining several topics in Pre-calculus. The algebraic function of f and g in *Figure* 7 can then be evolved about x-axis or y-axis to produce realistic surface of an object.

It is much more realistic and fruitful to utilize 3D animated images to elaborate on theoretical concept as shown in *Figure* 8*b*. 3D animations can make images look more realistic. Once the object is modeled, textures and colours can be applied to it. Lights and cameras can also be placed to give the 3D object some shadows and shadings(Mai, 1997).

Mathematics in Smart School

The concept of Smart School is growing rapidly which emphases the use of IT in all disciplines including Mathematics. Graphical visualization and Mathematical visualization have become an important tool in several fields, helping to understand mathematical concepts. With the aid of multimedia technology, self-accessed, self-paced and self-directed learning can be practised(Smart School, 2002).

Mathematicians can now use suitable software in Smart School to generate pictures that would be tedious or impossible to generate by hand in the past or were time consuming. However, the process of approximating a picture projected on a computer monitor is nontrivial.

The introduction of *e*-book will not help much in this area and particularly, the *e*-book concept for Mathematics. The full scale use of technology is beyond reach yet. A practical approach would be to train the teacher to utilize an affordable software such as Microsoft Excel. In this paper, most of the graphs are drawn using Microsoft Excel. The authors have refrained from using any CAS in producing the graphs but full colour 3D objects such as the "Pelita" done using Maple V.

Teachers in school/colleges should be provided with a PC and minor retraining at least once a year to get the latest knowledge in IT and IT applications in Mathematics. In times, when the hardware and software becomes affordable for every other students then it will be ripe to migrate the same techniques utilizing a bigger and more powerful mathematical engine. In the mean time, the teaching or the mode of delivery shall still be done manually through group activities and experiential learning mode.

Conclusion

The conclusion of this paper is that the real power of computer graphics lies in its ability to accurately/approximately represent objects of interest within its limited resolution. To understand how significant these features are, consider this: our entire common-sense knowledge of the physical world exists only in our mental models and have developed, through interaction, a mental picture that enables us to predict with some accuracy what is physically reasonable(Hanson et al., 1994).

Computer visualization has broadened the horizon of teaching in entire Mathematics. There are many activities, which can be observed in the area of producing graphical animations based on Mathematical concepts. In our opinion, time has come to upgrade the teaching of modern Pre-Calculus course such as Additional Mathematics. There should be made lean, lively group activities. The mode of teaching can make full use of technology and treat most topics graphically, numerically, and analytically. It will benefit the students not only at the upper secondary education level but years to come when they enter higher educational institutions.

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