# The Effect of Integrated Learning of Functions Using Computers ~As Quantitative Literacy in a Two-Year College Program~

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**Abstract:** In this paper we present a case study for exploring how integrated learning of functions by computer (Kakihana et. al, 2000) develops "function sense" (Fukuda et. al, 2001). Eight two-year college students were given a problem - solving task to perform in a computer lab based on the 1993 NAEP assessment (Dossey, 1993). Spreadsheets and websites were used. Surprisingly, students did not understand that numeric data in a table are representative data for a function. We also found that students did not relate a graph on a computer screen with an expression. Fortunately, we found that if students were competent in "function sense", they were able to solve a problem using computer technology even if they had been weak in mathematics at school. It was evident that development of "function sense" depends on how to juggle multiple representations (Goldenberg, 1997).

**Keywords**
function sense, quantitative literacy, liberal arts college students, technology, spreadsheet

## 1. Background

These days, all kinds of quantitative information surround us in our daily life about which we have to make judgments depending on the situation. Therefore, even students who are not majoring in science, engineering or mathematics must be able to interpret or represent mathematically these various types of information. Many researchers have discussed what the goal of mathematics should be and what should be taught in the mathematics curriculum for these students. Bennet and Brigges (1999) proposed that students need mathematics for college, for their career, and for survival in everyday life. As for our eight students, for one of them mathematics was necessary for her college entrance examination, for another it will be important in her career as a system engineer's assistant and for the others mathematics is basic to the survival skills required to understand the issues that confront them in daily life. Ever since Steen discussed quantitative literacy as mathematics for all people (Steen, L.A.,1997), "quantitative literacy" has become a common term in the teaching and learning of mathematics in liberal arts colleges. Furthermore, Bennet and Briggs (1999) describe how one of the ways for these students to achieve their goals in studying mathematics is to learn how to create and interpret graphs and models. "Function sense" (Fukuda C. et. al, 2001) would be a basic sense for these goals and an integrated learning of functions is an effective approach to achieving it. An integrated learning of functions refers to an integration of algebraic expressions with real world situations and to an integration of numeric data with various kinds of functions using expressions and graphs in function learning. "Function sense" has three components: to discuss and make judgments from real world numeric data (included in number sense (NCTM,1989; Sowder, 1992)), to make models for a situation and present them using functions (included in symbolic sense (Fey, 1990; Arcavi, 1994)), and to present functions on a graph or interpret graphs (included in graph sense (Freil S. N. et. al., 2001)). This study aims to determine the effects of developing "function sense" on solving a problem in an integrated learning of functions as well as to demonstrate how technology helps to solve a problem reflecting "function sense".

## 2. What is "function sense"?

"Function sense" refers to the area where number sense (NCTM, 1989; Sowder, 1992), symbol sense (Fey, 1990; Arcavi, 1994), and graph sense intersect (Freil, 2001)(Fig.1). Those senses represent certain ways of thinking rather than bodies of knowledge that can be transmitted to others (Friel, ibid. p145). For example, with "function sense", you could think of a function - such as the formula $y=3x+7$ - as an expression of some phenomenon and you could understand the difference of behavior from $y=3x^2+7$ (a part of symbol sense). You could also think of it as the straight line in a graph (a part of graph sense). Or you could imagine the sequence number 7,10,13,....from this expression (a part of number sense) (Fukuda et. al, 2001). Thus, one function can be approached in different ways and various kinds of functions can be used for a single everyday situation. In the school curriculum, linear functions, quadratic functions, cubic functions, and so on are taught separately and students usually feel that they have not digested any of them (Steen, 1990, p.4). In addition to this, the teaching of arithmetic and algebra in school has as its primary goal the training of students to manipulate numerical and algebraic symbols (Fey, 1990, p.62). These factors probably explain why many students find the learning of functions difficult and therefore not to their liking. Consequently, it is convenient to discuss the various kinds of functions in a single real-world situation by using a graph and a numeric table in order to develop "function sense" and competency in understanding functions. We have previously called this approach, an integrated learning of function (Kakihana et. al, 2000). Fukuda et. al. (2001) categorized "function sense" as numerical sense (included in number sense), visual sense (included in graph sense) and algebraic sense (included in symbolic sense). Behaviors reflecting these senses are listed as follows:

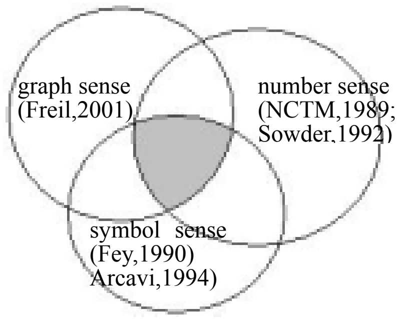


Fig.1 Function sense

For numerical sense of connecting numeric data in a table or graph -

A) To recognize quantitative relationships in concrete situations.
B) To identify variables and arrange data by making a table or graph -
C) To find an algebraic expression displayed visually to fit numeric data of two variables.

For visual sense of connecting a graph with features of expressions or data -

D)To scan a graph and interpret a verbally stated condition or find an algebraic expression.

For algebraic sense of connecting expressions with patterns of graphs or data -

E) To differentiate behaviors depending on degree of functions.
F) To check the result and judge the likelihood that it has been performed correctly.

## 3. Method

Eight two-year college students were given a problem to solve in a computer lab based on the 1993 NAEP (National Assessments of Educational Progress) assessment (Dossey, 1993). We tested the effects of developing "function sense" when solving a problem in an integrated learning of functions environment. We also listed how computer technology helps students to solve a problem. Through the teacher's observation, students' worksheets and interviews, we analyzed student activities from the perspective of behaviors reflecting "function sense" (Fukuda et. al, 2001). Concretely we analyzed behaviors as follows: (A - F correspond to the A - F above.)

A) To distinguish the shortest distance and distance along a road.
B) To make a table or graph on paper or on a spreadsheet.

C) To find algebraic expressions for this situation.
D) To scan a graph and interpret a verbally stated condition or explain the algebraic expression.
E) This behavior is not applicable to this task.
F) To explain the expression and the point P.

**Task**

Solve the following problem:

The darkened segments in figure 2 show the path of an object starting at point A and moving to point C at a constant rate of 1 unit per second. The object's distance from point A (or from point C) is the shortest distance between the object and the point.

1) Sketch the graph of the object's distance from point A and from point C .

2) Construct algebraic expressions.

3) Label the point where the distance of the object from point A and from point C is equal as point P.

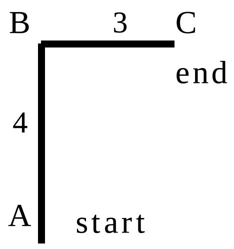



Figure. 2

This task is based on the 1993 NAEP in mathematics for 12th grade. Its purpose is to measure algebra and functions (http://nces.ed.gov/nationsreportcard/itmrls/qtab.asp). Students solved this problem referring to four kinds of functions (two linear functions and two irrational functions) and using a table and a graph during one class hour in a computer lab.

## 4. Results

The result of the analyzed behaviors reflecting "function sense" are shown in table 1.

Table 1. Results of analyzed behavior reflecting function sense

| | Miyuki | Miya | Yasue | Hiroko | Tomoka | Miho | Mariko | Akiko |
|---|---|---|---|---|---|---|---|---|
| math ability / Behaviors | above average | average | average | average | average | below average | low | low |
| A | ○ | ○ | ○ | ○ | × | ○ | × | ○ |
| B | ○ | ○ | ○ | × | × | □ | × | ○ |
| C | ○ | □ | □ | × | × | □ | × | □ |
| D | ○ | ○ | ○ | ○ | × | □ | × | × |
| E | | | | | | | | |
| F | ○ | ○ | ○ | ○ | □ | ○ | × | □ |

○:complete □:partially incorrect ×:incorrect

The results of each case are as follows:

**1) The Case of Miyuki** : She completed this task perfectly. She will need mathematics for her career and she knows the basic relation between a graph and a function and also remembered the Pythagorean theorem. She indicated the zero to four-second points with ease and used the Pythagorean theorem to calculate the other 5, 6 and 7-second points. She used the square root function in a spreadsheet to calculate the distance and plotted each point on a paper connecting them with a straight line. At the same time, she understood that the line between each point is not straight. She did not know how to draw the graph for the function $y=\sqrt{4^2+(x-4)^2}$ , but she could draw the graph using points and describe the behavior of this function as follows:

Teacher: "Is the line between the two points straight?"

Miyuki: "No, it is not."

Teacher (*pointing to the formula* $y=\sqrt{4^2+(x-4)^2}$ ): "Do you know how to draw the graph for this function?"

Miyuki : "No, I don't. But it must be almost the same shape as this curve."

Teacher: "How do you know that?"

Miyuki : "I'll add more points to be calculated."

She completed this task perfectly from the viewpoint of "function sense". She was able to explain the point P as the intersection of the two functions. She recognized quantitative relationships in concrete situations and identified the two variables as time and distance and made a table and graph. She also found algebraic expressions for each situation and checked the result and judged it likely she had performed the task correctly.

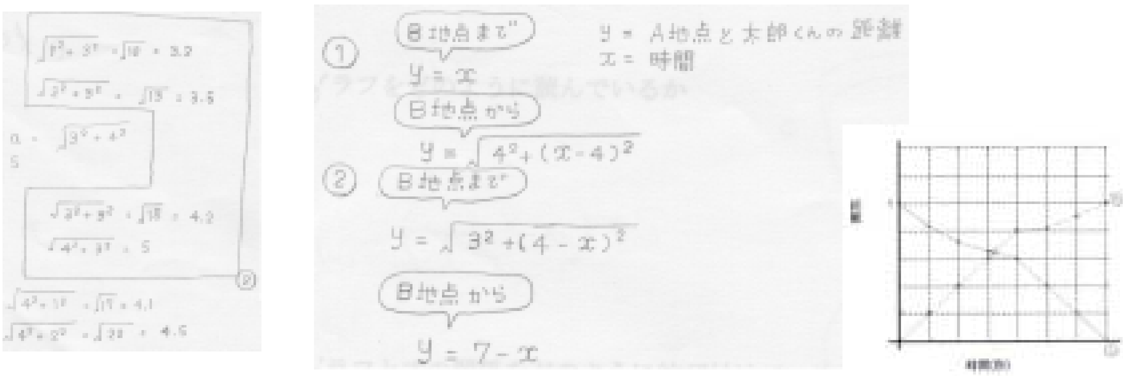



Fig. 3 Miyuki's calculation, expression and graph

**2)The case of Miya:** She relied on numeric data rather than a graph to correct her expression. She checked her graph with a graph drawn by technology.

She calculated from 0 to 4 second points in her head and connected each point. Then she tried to measure by scale but because the length on the figure is not the true one, she gave up and then calculated 5, 6 and 7 second points by using the Pythagorean theorem. She used the spreadsheet to calculate the square root and to confirm her graph.

From her graph she constructed an expression y=-x.

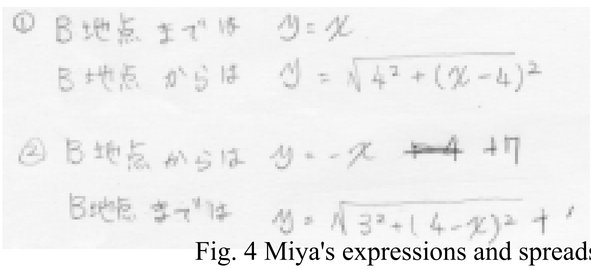
Teacher: "How did you construct this expression?"

Miya (*Pointing to the formula y=x* ) : "This part is the opposite direction of this graph."

Teacher (*pointing to the formula y=-x*): "Is this expression correct for this line?"

Miya (*After substituting 5 for x*): "No, it is not correct. +4? No. +5, No." Then at last she exclaimed, "Ah! It is y=-x+7."

Her approach is an appropriate one to develop "function sense" because she used numeric data to find the correct expression and connected them with a graph. She recognized quantitative relationships in concrete situations and identified the two variables as time and distance and made a table and graph. Her work is almost complete from the point of "function sense" .




| | |
|---|---|
| 0 | 5 |
| 1 | 4.24 |
| 2 | 3.6 |
| 3 | 3.16 |
| 4 | 3 |
| 5 | 2 |
| 6 | 1 |
| 7 | 0 |

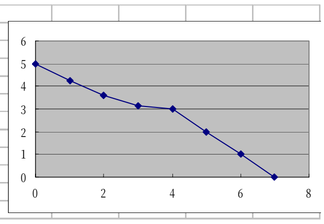


Fig. 4 Miya's expressions and spreadsheet graph

**3) The Case of Yasue:** She understands the basic relation between a graph and a function and also remembered the Pythagorean theorem. She calculated from 0 to 7 seconds and connected each point. She used a spreadsheet to calculate a square root and confirmed her graph on paper. As the expression from 5 to 7 started from point C, she also wrote "y=-x" as an algebraic expression.
Teacher: "How did you construct this expression?"
Yasue: "From 5 it is decreasing one by one."
Teacher: "Is this expression correct for this line?"
Yasue (*looking at the graph again*): "No, it is not correct." (*Expanding the line and finding an intersection with the y-axis*) "Ah! It is y=-x+7."
She recognized quantitative relationships in concrete situations and identified the two variables as time and distance and made a table and a graph. She used the graph to correct the expression. Her work is also almost complete from the point of "function sense."

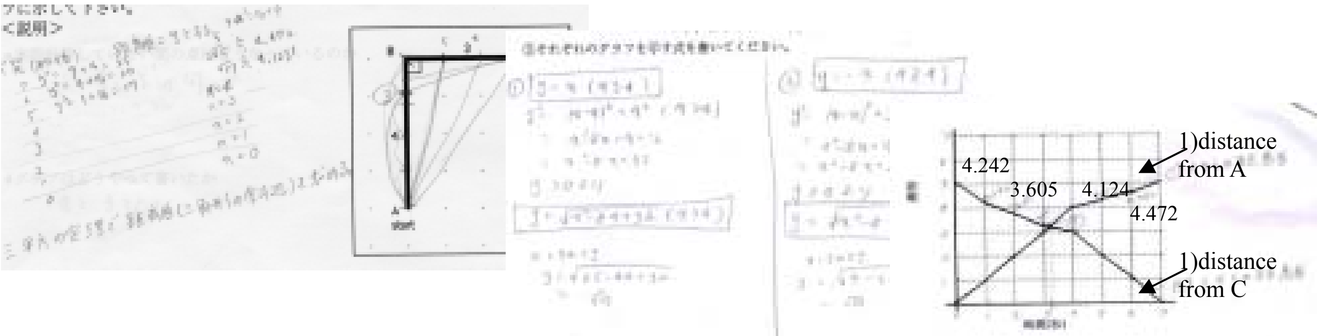



Fig.5 Yasue's worksheet and graph

**4) The Case of Akiko:** She dropped out of mathematics in her middle school. She did not remember the Pythagorean theorem and the behavior of the function y=ax+b. Despite this she was able to complete her task almost correctly using a computer. She recognized the concrete aspect of the quantitative relationship in the task. At first she tried to use the trigonometric ratio because it is a hypotenuse of a right triangle and also to find formulae on a Webpage but realized they were not appropriate for this task.
Teacher: "Do you remember the Pythagorean theorem?"
Akiko: " No!!"
As soon as she heard the teacher's comment she searched for the Pythagorean theorem on the Internet. However the first page she accessed dealt with the proof of the theorem. As she could not understand this, she searched the web again. She found a page where the Pythagorean theorem was shown in a figure and understood that she could apply this theorem to the task (Fig.6). She input this formula in cells on a spreadsheet to obtain a table and a graph.

$a^2 = b^2 + c^2$　　$x$ = 時間

$y = \sqrt{(x-4)^2 + 4^2}$

| 秒 | 距離 |
|---|---|
| 0 | 5.656854 |
| 1 | 5 |
| 2 | 4.472136 |
| 3 | 4.123106 |
| 4 | 4 |
| 5 | 4.123106 |
| 6 | 4.472136 |
| 7 | 5 |

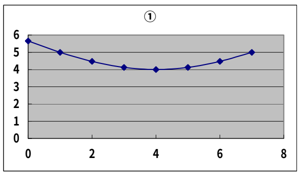



Fig.6 Akiko's expression and table and graph on a Spreadsheet

Akiko: "This graph cannot be utilized for seconds 0 to 4."

Teacher: "How did you draw this line?"
Akiko: "One is one, two is two, so I connected zero to zero and four to four. From four to seven, I input the formula into spreadsheet cells and constructed its graph (Fig. 6, right) and copied this section of the graph on to paper .
Teacher (*pointing to the formula* $y = \sqrt{4^2 + (x-4)^2}$ ): "How did you get this expression?"
Akiko (*pointing to figure 7*): "This part is x-4, so I substituted x-4 to this theorem ."(Fig. 6, left)
She did not write the expression for the parts of the straight line.
Teacher (*pointing to the straight line*): "What is the expression of these parts?"
Akiko: "Until 4, y= x."
Teacher: "How did you get that?"
Akiko: "When it is one second, one unit, when two, two seconds, so y=x."
Teacher: " How about this part?"
Akiko: "This part? It's in the opposite direction, so it is y=-x."

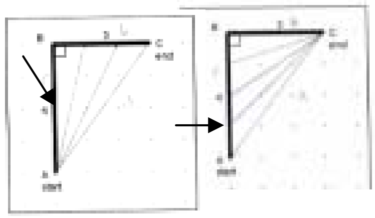
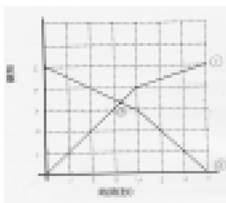


Fig. 7 Akiko's figure

Teacher: " Are you sure?"
Akiko: "Ah! No, this part is the rest of 7. So y=7-x."
Teacher: "Didn't you think about the expression from a graph? Do you know where the b of the expression y=ax+b is on a graph?"
Akiko: "???"
Teacher: "Have you ever heard of an intercept with the y-axis?"
Akiko: "Ah!, I might have learned something like that in junior high school. But I did not understand the relation between a graph and an expression in junior high school, anyway."

**5) The Case of Hiroko:** Although she remembered the Pythagorean theorem and the behavior of functions, she was unable to relate numeric data with the situation.
She calculated the distance for 5, 6 and 7 seconds using the Pythagorean theorem. She did not use these data to draw a graph. She drew a graph to connect (0,0) with (4,4)) and (4,4) with (7,5) using a straight line(Fig. 8).
Teacher: "Why did you not use these data to draw a graph?"
Hiroko: "These data include the square root, so I never thought to use these data to draw a graph."

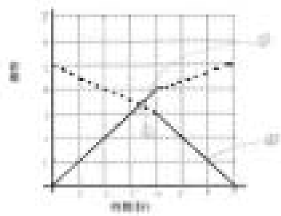


(1) $x \le 4$
y=x
$4<x$
$y = \frac{1}{3}x + 2\frac{2}{3}$
(2) $x \ge 4$
$y = \frac{1}{2}x + 5$
$4<x$
y=-x

Fig. 8 Hiroko's graph and expressions

Teacher: "How did you get these expressions?"
Hiroko: "From a graph."
Teacher: " Is y=-x a real expression of this line?
Hiroko: "Oh! No the line on this graph will intersect the y-axis at 7. Therefore y=-x+7."

**6) The Case of Miho:** She remembered the Pythagorean theorem and calculated the distance for each second and input these data into a spreadsheet and made a graph. When she draw the graph on paper, she wondered, "How can I draw this curve?" She drew a straight line with dots to express a curve (Fig. 9). She knew this graph was different from a straight line but was unable to connect the graph on the computer screen with that on paper. She also could not relate the expression and the graph.



$x^2=4^2+(\text{fromBtoC})^2$
$x^2=3^2+(\text{fromAtoC})^2$

Fig. 9 Miho's graph and expressions

**7)The Case of Tomoka and Mariko:** Both said, "I can't do this task because it doesn't say how to solve it and doesn't provide any formula with which to solve it." So from the very start they made no attempt to solve it. When they saw their friends starting to search for a formula on the Internet, they tried to use the computer too but gave up because they still did not understand what to do.

## 5. Discussion

Students in liberal arts or two-year colleges tend for the most part to feel no need for mathematics in their life and simply take mathematics courses for graduation credits. As mathematics teachers, therefore we don't expect our students to learn any special mathematics knowledge. Instead, we expect our students to be able to read quantitative data and graphs appropriately and to make correct judgments when confronting situations that utilize quantitative information. Consequently we wonder what kinds of mathematics we should teach them. Bennet and Briggs (Bennet & Briggs, 1999) identified four areas of mathematics for non-scientific students: 1. Logic, Critical Thinking, and Problem Solving 2. Number Sense and Estimation 3. Statistical interpretation and basic probability 4. Interpreting graphs and models. For each of these areas, "function sense" is the basic sense. How to teach and learn functions has long been the subject of discussion and many terms for learning function have appeared in mathematics education. For example, in Nine Year Book (NCTM,1934), we see "function thinking" and "functional relation". In Japanese, also, there are many terms; for example "Kansu Shiso (function thinking)" (Wada,1953) and in the New Edited Dictionary for Arithmetic Education and Instruction (Japan Society of Mathematical Education, 1993), we see "Kansu no Kangae (thinking of functions)", "Kansu Kankei (functional relation)" , "Kansu tekina mikata (a way of thinking about related functions)" and so on. While it is very hard to differentiate precisely between these words, all of them expect students to learn the relation between numeric data while paying attention to correspondence and changes in these data. For this goal, students have to be interested in a changing phenomenon and select numerical data to investigate it (Shimizu, S, 1998). Despite this, linear functions, quadratic functions, cubic functions and so on are taught separately and formally in the school curriculum. Students are taught the relation between a phenomenon and symbolic expressions very occasionally. Instead they spend most time manipulating numerical and algebraic symbols. As technology has developed, the environment of learning functions has been changing. Multiple representations linked by a string of symbols such as $f(x)=x^2-3x+4$, tables, graphs, specialized icons, and verbal description (Goldenberg, 1997) are now available. Accordingly, it is time to revise methods for learning functions and solving tasks. We proposed an integrated learning of functions and identified "function sense" as an ability to be developed in learning functions. Goldenberg stated multiple representations as a vehicle for understanding Understanding. The task based on NAEP in this study was appropriate to analyze the behaviors reflecting "function sense" in an integrated learning of functions and highlighted how students juggle multiple representations.

**A) With "function sense" students can solve a problem with the aid of computer technology even if they have very little mathematics knowledge.**

As the result of the analysis of students' activities from the viewpoint of behaviors reflecting "function sense", students who had both high school mathematics knowledge and a "function sense" like Miyuki could apply their knowledge to solving a problem. Students like Hiroko and Miho, who had a little mathematics knowledge but no "function sense" could not relate the numeric data and a graph and could not relate a graph on a spreadsheet and on paper. A student like Akiko, who had very little mathematics knowledge but has "function sense" could solve a problem using computer and relate the numeric data with a graph and an expression. In particular, she was able to recognize quantitative relationships in concrete situations and identify some variables and arrange data by making a table or a graph. This ability may result from her often utilizing a computer to gather information, analyze quantitative data in her daily life and construct graphs for these data. Akiko can also check the result and judge the likelihood of it having been performed correctly. When she forgets a formula for solving a problem, she never hesitates to search for it on the Internet and displays an ability to judge whether it is useful or not.

**B) In an integrated learning of functions, students could construct an expression in various ways.**

In the case of Miya, Yasue and Akiko, they mistook the expression for the straight line. They imagined that the graph must be in the opposite direction because the path from C is in the opposite direction from the starting point A. Then they could correct their answer in their own ways. Miya substituted a couple of numbers in the expression and checked the value on a graph. Yasue observed the graph and applied her knowledge of the linear function. Akiko did not understand the feature of a linear function but she remembered how she calculated the distance for each second and applied it to construct the expression. She had very little chance to observe graphs for various kinds of functions when she was in middle school, so she was unable to imagine being able to construct the expression from a graph. Yet she is able to recognize quantitative relationships in concrete situations. Miya and Yasue identified some variables and arranged data by making a table or a graph. While Akiko cannot relate a graph and expression, she could relate a graph drawn on a spreadsheet to a concrete situation.

**C) A spreadsheet graph was not connected with a graph written on paper.**

These days spreadsheets are very popular in the computer environment and are considered very useful for viewing numeric data and a graph simultaneously. Spreadsheets were, however, originally designed for uses outside the classroom (Yerushalmy, 1999). Therefore, we have to be mindful about how students recognize these graphs when we use spreadsheets for education. Sometimes students do not relate data in a table with those in a graph as in Miho and Hiroko's case. For students like Akiko and Miho, however, we have a case in which there is inability to relate an expression to a graph. Students construct graphs very easily form numeric data but find it hard to relate a graph and an expression.

## 6. Conclusion

The results of this study show that students surprisingly did not understand that numeric data in a table are representative data for a function and that students did not relate a graph on a screen with the expression. Therefore, when using spreadsheets in class, teachers should emphasize the relation of numeric data in the function as well as the relation of the graph and function. The students juggled with the interaction among representations in an attempt to understand the bigger picture as Goldenberg also observed in this studies (Goldenberg, 1997, p155). Fortunately, we found that if students had an ability of "function sense," they could solve a problem using computer technology even if they had been weak at mathematics in school. In an integrated learning of functions, multiple representations linking symbol expression, table, graphs, specialized icons, verbal description, and physical situation are available. How students juggle these multiple representations leads to "function sense" and it is up to teachers to create tasks for an integrated learning of functions to develop it.

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