A Method to Find Calculating Errors Based on Misconceptions at an On-line Exercise System

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Abstract

Although our on-line exercise system of symbolic calculation supported average students well to improve their manual calculating skills, it hardly helped the slow learners, whose errors often came from their misconceptions. The feedback information returned by the system was less valuable to them, because it only included the position of the error in the calculating steps but not the type or cause of the error. The slow learners were not able to correct the errors with the help of the simple information.

The slow learners have several misconceptions in symbolic calculations, and they repeat the same types of errors in their calculations. They repeat the errors because their calculations are *similar* to the correct calculations based on mathematical rewriting rules. They would not change the method if the system does not point out the misconceptions properly and let them convince.

In this paper, we describe a method to find the errors caused by typical misconceptions often observed in the calculations of rational expressions done by slow learners, and to let them recognize their misconceptions. It uses MATHEMATICA, a mathematical programming language, and is implemented as additional functions into the WWW-based on-line exercise system.

1. Background

For the effective computer supported exercises of symbolic calculations, rich feedback information in return to the students' inputs is essential. Some systems are known to return richer information than correct / incorrect of the final answers, for example, certain comments to the intermediate expressions on the way of the calculations [1, 2, 3, 4].

The experience using such a system shows that the positions of errors in intermediate calculating steps are valuable information to average students, who understand the calculating method but lack in the amount of practice. Once the positions of errors are shown, they easily find the causes of the errors because the errors are usually careless mistakes and merely the result of their immature calculating skills. The system helps them a lot to improve their calculating skills.

On the other hand, some slow learners freeze in front of the same information. Even the two expressions before and after the incorrect calculating step are given, the information does not make an effective hint for them because most of their errors are not careless mistakes but the natural results deduced from their misconceptions. If the system fails to let them convince through its compact explanation, they openly appeal the validity of their calculations to the teacher. More detailed explanation why and how their calculations are incorrect is necessary.

Interactive step-by-step instruction [5, 6] is one of the methods to guide such slow learners in the calculation of rational expressions. The system puts forward only one step of the calculation according to a student's instruction that states a part of the expression to be rewritten and declares a calculating rule selected from a list. Because the list has only correct rules, the calculating step is guaranteed its mathematical validity. Using the system, the student decides what kind of a rule he/she is to apply when he/she puts forward a calculating step, and is expected to get familiar with the calculation based on mathematical rules.

The weakness of the method is its loose connection with exercises and passive role of the students who use it. It is possible for a student to complete the instruction course without learning any calculating rules. The student might learn the operations in the instruction page, such like which a calculation rule to select for each calculation, and complete the course. But even after that he/she wound not pay any attention to the meaning of the calculating rules. In such a case, the student would continue to make errors based on his/her misconceptions even after the completion of the course.

It is the purpose of this paper to propose another method to guide such students. First step is to find the incorrect calculations lead by misconceptions often found in their calculations of rational expressions. Second step is to give the students additional explanations of calculating errors using numerical evidence. The evidence poses a conflict with the students' belief.

2. Typical Errors Based on Misconceptions and the Common Features

Table 1 Calculating Errors Based on Misconceptions

LHS	RHS	Error Type	
$\dfrac{(x+1)^2-(x-1)}{(x+1)(x-1)}$	$\dfrac{x+1-(x-1)}{x-1}$	(A) Incorrect cancellation	$\dfrac{A\cdot C+D}{A\cdot B}\neq\dfrac{C+D}{B}$
$\dfrac{(x+1)+x}{(x+1)-x}$	$\dfrac{x}{-x}$	(A) Incorrect cancellation	$\dfrac{A+C}{A+B}\neq\dfrac{C}{B}$
$\dfrac{x^2+2xy+y^2}{x^3+y^3}$	$\dfrac{2xy}{x+y}$	(A) Incorrect cancellation	$\dfrac{A\cdot C+D}{A\cdot B}\neq\dfrac{C+D}{B}$
$\dfrac{x^3}{x+\dfrac{x}{x^2+1}}$	$\dfrac{x^3}{x}+\dfrac{x^3(x^2+1)}{x}$	(B) Incorrect addition of fractions	$\dfrac{C}{A+B}\neq\dfrac{C}{A}+\dfrac{C}{B}$

Table 1 shows some example errors based on misconceptions found in the calculations of rational expressions done by first grade students (ages 15 – 16) of our department in 2002/2003 school year. Theses errors used to be found hardly among the answer sheets of the students, have come to appear in the recent several years, and are regularly observed in their answer sheets nowadays. They are apparently incorrect for teachers, but not so to the students.

Common features of those students and their mistakes are listed as follows.

1) They have enough experience in and are good at numerical calculations of fractions,

2) But they have less experience in the algebraic fractions.

3) They often calculate as if they ignore the mathematical rules, and simply delete common symbols or numbers both from the numerator and the denominator without paying any attention to the structures, for example. They ignore the priority of operations and add the denominators of two fractions to make the denominator of a common fraction.

4) Such ignorance of rules is most often observed in the calculations of rational expressions.

5) They have been taught the right calculation rules in classes, but they seem to forget what they have learned easily and tend to repeat the similar errors time to time afterwards in the learning of other subjects.

The interview to some of the students revealed that their knowledge about calculation was not logically structured and they strongly depended on patterns rather than mathematical rules in their calculations. For example, a basic idea of their cancellation was to delete common numbers or symbols both from the numerator and the denominator of a fraction. It came from their experience of simplifying numerical fractions, where only a single number always represents the numerator and the denominator. They actually used common factors to simplify numerical fractions, but they did not recognize them as factoring. So, it was natural for them to delete common symbols without factoring the numerator and the denominator when they faced an algebraic fraction that had more complex structure of symbols, numbers and operators in the numerator and the denominator. In such a calculation, students were to ignore the fact that a fraction could only be cancelled by a common factor in the numerator and the denominator.

They could not explain why they ignored the priority of operations in their calculations. Although they knew the basic priority, multiplications first and additions second, and follow it in numerical calculations, they easily forgot the same rule in symbolic calculations, especially in rational expressions. Some said that the equality is obvious in numeric calculation but not so in symbolic calculation. A simple method is needed for them to consider the priority of mathematical operations.

In summary, their calculations were based upon matching patterns rather than calculation rules, and they tended to prefer memorizing solutions directly than understanding the underlying mechanisms [7]. Matching patterns might work well in numeric calculations, but not in symbolic calculations. What those students must learn in their calculation is reasoning. Calculations of rational expressions should be a good opportunity for them to learn the attitude to think and pay more attention to calculating rules or the mechanism of calculations.

3. A Method to Find the Errors Based on Misconceptions

Most CAS (Computer Algebra System) have build-in functions to judge the equality of two symbolic expressions, and our on-line exercise system, which uses MATHEMATICA as the calculating engine, uses such a function in finding calculating errors in the students' calculating steps. But a CAS does not possess any functions to separate an error based on misconception from the other errors. It is not the nature of a CAS to handle any incorrect calculations. So, we need to develop a custom function to achieve that.

The method proposed in this paper is a result of the observation of typical errors done by our students as shown in Table 1, and tries to simulate their calculating process as much as possible. For example, it tried to detect an incorrect cancellation as shown in Table 2.

Table 2 Example Procedure of Detecting an Error (Incorrect Cancellation)

	LHS	Equality	RHS
Original expression (a student's calculation)	$\frac{x^2+2xy+y^2}{x^3+y^3}$	$\neq$	$\frac{2xy}{x+y}$
Original numerator	$x^2+2xy+y^2$		$2xy$
Replaced numerator	$x^2\cdot 2xy\cdot y^2$		$2xy$
Original denominator	x^3+y^3		$x+y$
Replaced denominator	$x^3\cdot y^3$		$x\cdot y$
Replaced expression (for detection)	$\frac{x^2\cdot 2xy\cdot y^2}{x^3\cdot y^3}$	$=$	$\frac{2xy}{x\cdot y}$

In the example, the original expressions, LHS (Left Hand Side means the expression before the calculating step) and RHS (Right Hand Side means the result of the calculating step), are not equal to each other. We suspect that the student tried to cancel x^2 and y^2 from the numerator and the denominator, and he simply deleted them. Because the denominator had x^3 instead of x^2, he left x^1 there. This is a typical example of their incorrect cancellations.

Detection of errors should follow the behaviors of the students that they mix additions with multiplications and ignore the priority of the operations. If we choose the numerator of the LHS ($x^2+2xy+y^2$) and replace the main operator *Plus* with *Times*, the replaced numerator becomes $x^2\cdot 2xy\cdot y^2$. And after the replacements at all the numerators and the denominators in LHS and RHS, the replaced expressions become equal.

We can say it a calculation based on a misconception " if two non-equivalent expressions, LHS and

RHS, become equal when we replace the main operator of the numerators and the denominators from *Plus* to *Times*.” The main operator is the first operator in functional notation and makes the root of a tree structure as shown in Figure 1.

Algebraic notation : $x^2 + 2xy + y^2$

Functional notation :

Plus[Power[x,2], Times[2,x,y], Power[y,2]]

Structure :

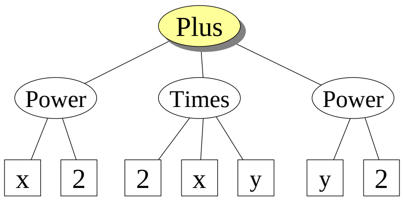

Figure 1 The structure of an algebraic expression

The method could be applied to detect another typical error, “ incorrect addition of fractions” and their reverse calculation (Table 3).

Table 3 Example Procedure of Detecting an Error (Incorrect Addition of Fractions)

	LHS	Equality	RHS
Original expression (a student’ s calculation)	$\dfrac{x^3}{x+\dfrac{x}{x^2+1}}$	$\neq$	$\dfrac{x^3}{x}+\dfrac{x^3\left(x^2+1\right)}{x}$
Remove GCF from numerators	$\dfrac{1}{x+\dfrac{x}{x^2+1}}$	$\neq$	$\dfrac{1}{x}+\dfrac{x^2+1}{x}$
Original numerator	1		$\dfrac{1}{x}+\dfrac{x^2+1}{x}$
Replaced numerator	1		$\dfrac{1}{x}\cdot\dfrac{x^2+1}{x}$
Original denominator	$x+\dfrac{x}{x^2+1}$		1
Replaced denominator	$x\cdot\dfrac{x}{x^2+1}$		1
Replaced expression (for detection)	$\dfrac{1}{x\cdot\dfrac{x}{x^2+1}}$	=	$\dfrac{1}{x}\cdot\dfrac{x^2+1}{x}$

In this case, the GCF (Greatest Common Factor) of the numerators of all the fractions in RHS and LHS must be removed from the expressions before the replacement of the *Plus* operators.
As shown in two examples, the method is effective to separate typical errors based on misconceptions from the other errors like careless mistakes. Finding the errors based on misconceptions gives the student richer information along with the finding of careless mistakes [8]. The separations of errors make it also possible for the exercise system to react to each error differently depending upon its type.

4. Explanation of the Misconceptions

Our on-line exercise system is composed of a WWW server connected to students' terminal computers through the computer network. The server has a WWW server program, a CAS (MATHEMATICA), and a connecting mechanism (WebMATHEMATICA). The interface is described basically in HTML, and the evaluating logic is written in custom functions coded with MATHEMATICA programming language. The new method is added to the system as custom MATHEMATICA functions.

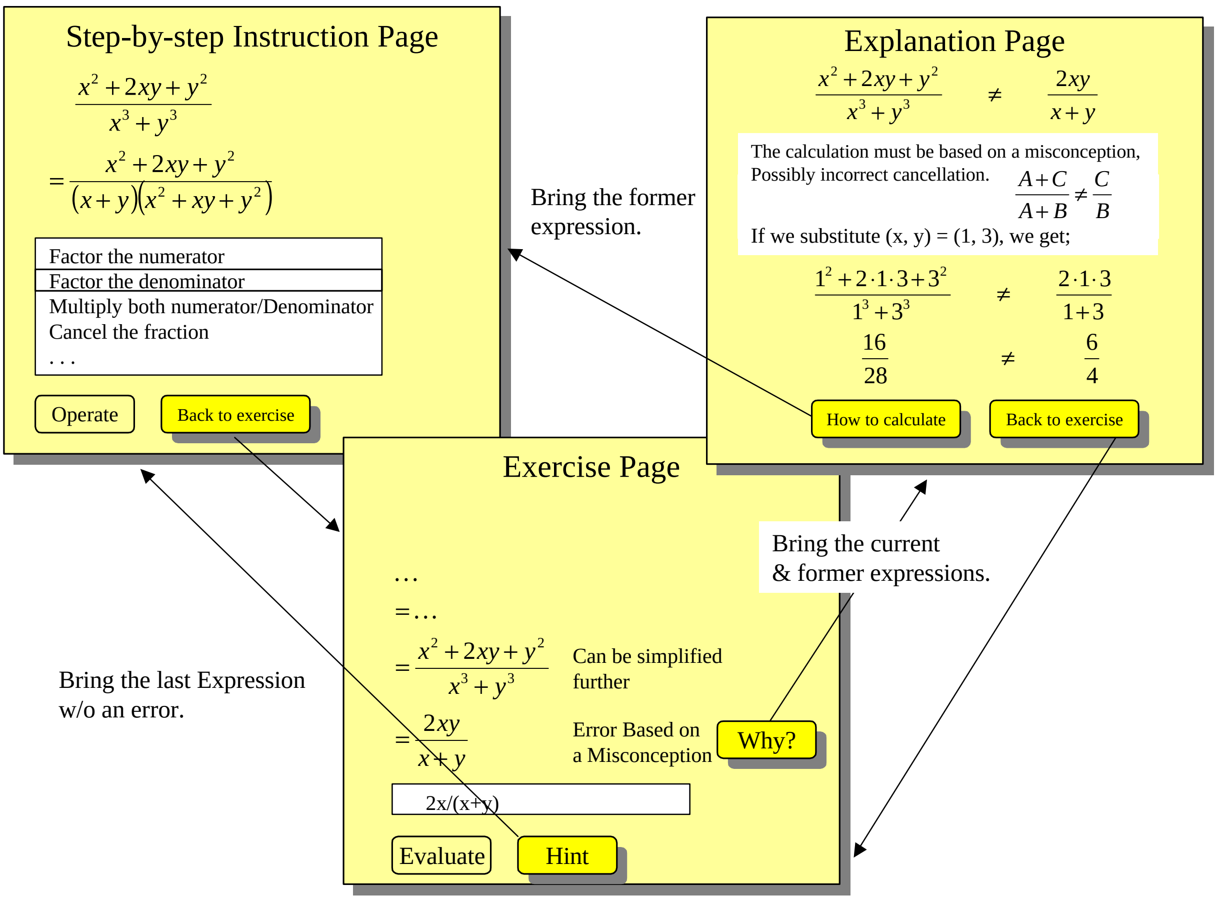

Figure 2 Relation of the pages

The system has three types of pages: an exercise page, a step-by-step instruction page, and the new explanation page for misconceptions (Figure 2). The exercise page serves the main service of the system. In the page, every intermediate expression on the way of a calculation receives a comment from the system. The comments describe how far the intermediate expressions from the expected final answer, point out the calculating error if the expression is not equal to the answer expression, and tell existence of a certain misconception in the calculating error.

An error caused by a misconception also provokes an action on the system side: adding a hint button linked to the explanation page or directly displaying the explanation page over the exercise page. The former action is for rare errors and the latter for more frequent errors.

The explanation page displays two expressions before and after the calculating step in the exercise page, the description that the calculation includes an error based on a misconception, and a pair of numerical expressions to show the inequality. The numerical expressions are made simply substituting selected numbers for the variables in the symbolic. The numbers are smallest positive integers, which avoid to make zero denominators or to suffice the equation RHS = LHS.

After reading the explanation, the students have options to straightly return to the exercise page or to move to the step-by-step instruction page to learn the right calculation rules.

5. Discussion

The detection of an incorrect calculation is possible when the calculation includes only a single step. The incorrect calculation shown in Figure 3 is not detected by the method because it includes two steps, incorrect cancellation and simplification of the denominator.

LHS RHS

$$\frac{x^3\left(x^2-1\right)}{x\left(x^2-1\right)+x} \quad \left(=\frac{x^3}{x+x}\right) \quad =\frac{x^3}{2x}$$

Incorrect cancellation simplification of the denominator

Figure 3 An example of the incorrect calculation that can not be detected with the method

We think the method is effective even with this limitation if teachers encourage students to describe intermediate calculating steps in detail. Many teachers must agree that describing calculating steps in detail is effective in the learning of mathematics.

Showing numerical expressions as a counter-example is a convincing explanation. It is especially effective when the students are good at numerical calculation enough to confirm the inequality of two numerical expressions immediately. They are fast to realize the contradiction between their

symbolic calculation and the numerical inequality, and admit that their calculation must have an error.
We think the system is effective to change the students' attitude from matching patterns to calculating based on rules through the exercises of algebraic fractions. If the students once gain the right attitude, they would not make similar calculating errors in the calculation of irrational expressions, exponential or trigonometric functions.

6. Conclusion

The authors implemented a new function into our on-line exercise system, which separates calculating errors based on some misconceptions often observed among slow learners. We collected the common features of the calculating errors from their answer sheets, and interviewed them to find the reasons. The common features lead us to develop a method to separate the errors based on misconceptions from the other errors. The method includes a convincing explanation using numerical example as the evidence.

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