# Mathematics That Makes Computers Possible

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## Abstract

The enormous advances in the development of computing machinery in the 20th century are attributed to the maturity of the underlying algorithms required to operate them. These algorithms provide the various functionalities that a computer is expected to perform under various environments on different platforms. Algorithms, in turn, are based on mathematical principles and the logical steps associated with it. We describe the development of mathematical ideas and concepts throughout the centuries culminating in the formulations of algorithms and the computing machinery whose existence depends so much on them. We went all the way back to Aristotle's attempt at logical thinking and we end with the ideal universal Turing machine. In the process we acknowledge the valuable contributions from mathematicians and logicians alike, and we argue that the future development of computing science depends very much on the selfless efforts of such professionals again.

## Introduction

Nobody can deny how much computers have affected our everyday lives, for good or bad. Today, computers do much more than simply compute: automatic teller machines (ATM) made transactions possible anywhere and anytime; scanners in department stores total up our purchases and at the same time, keep track on the stores inventory; and the technology of the smart card made it possible to use the ONE card for almost anything, as and identification card, charge card, library card, or even as cash. Who are behind these great discoveries? What drove them to do what they did, to think the way they did and to come up with a machine, which later on developed to be such a great tool?

No doubt a lot has been said about all aspects of computers. However, in this paper, we would like to highlight the mathematics without which computers would not be invented in the first place. To date, the relevant mathematics in computer science that have been identified are semantics of computation, algorithm and mathematical logics [Winskel]. Semantics of Computation answers the question what is computer science, Algorithmic describes how to carry out a task and mathematical logic gives proof systems to determine if the method is correct or not. We will try to highlight how these aspects developed from the time of Aristotle's statement of logic until the time Turing designed his machine. These mathematicians' contributions were very valuable and by highlighting them it is hoped that greater understanding and appreciation are achieved from people whose lives revolve in the constant presence of computers.

## Ancient Logic

It all started with Aristotle, the great philosopher. He was one of Plato's most influential and greatest students. His main contribution was in establishing logic as an instrument by which we get to know anything. In short, he believed that careful consideration of what we say can help us understand the way things really are.

Aristotle was born in 384 BC in Stagirus, Macedonia, Greece. He would have followed his father's footstep as a doctor if he had not lost him while he was so young. However his father's earlier lesson and his guardian's influences had made him good in Greek, rhetoric, poetry and biology. In 367 BC, at the age of 17, Aristotle went Plato's Academy, first as a student, then as a lecturer. At the end of his twentieth year there, due to the political development at that time, he went back to Macedonia.

In 335 BC, Aristotle opened up his own school, the Lyceum in Athens, with the help of several assistants and materials he brought from Macedonia. Most of his works, believed to be his lecture notes, were recovered after his death. One of his earliest works in logic was the "Categories". He proposed that the description of individual things that attribute to each predicate or category to be developed. There were to be of ten different sorts namely substance (the most important), quantity, quality, relative, where, when being in position, having, acting on and being affected by.

In "On Interpretation" Aristotle considered the use of predicates in combination with subjects to form propositions or assertions that can either be true or false. This however was met with problems regarding the truth or falseness of a proposition. For example a proposition that is true today can be false tomorrow.

Aristotle further produced "Prior Analytics", "Posterior Analytics", "Physics I" and "Physics II" in an attempt to explain why things happen the way they do, based on existing state and knowledge.

His famous statement is from Prior Analytics, known as Aristotelian Syllogistics[Kemerling]:

*Every Greek is a man*
*Every man is mortal*
*Every Greek is mortal*

These statements were a form of argument with two premises and a conclusion. Mathematicians, who had analyzed these statements since the 1920's, felt that this ancient logic revealed modern philosophical features. They also became the basis of later statements of logic. Aristotle died in 322 BC.

## Leibniz's Dream

*"To find a true alphabet of human thoughts and the appropriate calculation tools to manipulate these symbols"*

Gottfried Wilhelm Leibniz was born in Leipzig in 1646. His father is a philosophy professor, who died when Leibniz was six years old. By the age of eight, he had become a fluent reader of Latin, thank you to his father's library.

He obtained his Bachelor's degree at Leipzig, wrote a thesis on Aristotelian metaphysics. He was fascinated by Aristotelian division of concepts in Categories and thus was inspired to develop a system based on alphabet whose elements represented concepts. He obtained his Master's degree from the same university by concentrating on looking at the relationship between philosophy and law. With his deep interest in law, he pursued a second Bachelor's degree in this area, namely, the use of mathematical logic in dealing with the law. He obtained his Ph.D. at the age of 21 from the University of Altdorf .

He then moved to Paris in 1672 and was assigned to update the legal system by his wealthy patron, the Baron Johann Von Boineburg, Among other things, he had to deal with Boineburg's financial affair. In 1673, Leibniz exhibited his calculating machine in London, the first one which was able to carry out all the four basic arithmetic operations, namely, addition, subtraction, multiplication and division. This machine is a better one, compared to the machine designed by Blaise Pascal.

One of his goals is to have a philosophical language free of ambiguities and facilitates human communication and thus, to end the controversies with reasoning. His famous dream was to find a true alphabet of human thoughts and the appropriate tools to manipulate these symbols. The grand program needed to achieve this goal consisted of three major steps. The last one was the rules of deduction which he called *calculus ratiocinator* , or known today as *symbolic logic*. His attempt to produce this is shown below [Davis]:

> DEFINITION: A is in L, L contains A, is the same as to say that L can be made to coincide with the plurality of terms taken together of which A is one. $B \oplus N = L$ signifies that B is in L and that B and N together compose or constitute L. The same thing holds for a larger number of terms.
>
> <u>Axiom1.</u> $B \oplus N = N \oplus B$
>
> Postulate: Any plurality of terms, as A and B, can be added to compose a single term $A \oplus B$.
>
> <u>Axiom 2</u>. $A \oplus A = A$

Proposition : If A is in B and A = C, then C is in B.

Proposition: If C is in B and A = B, then C is in A.

Proposition: A is in A.

Proposition: If A is in M and B is in N, then $A \oplus B$ is in $N \oplus M$

This example of Leibniz's logical calculi was seen as an algebra of logic, where rules were specified for manipulating logical concepts. Some scholars saw Leibniz as one of the first individual who envisioned something that is close to today's artificial intelligence.

## Boolean Algebra

George Boole was born on November 2, 1815 in Lincoln, a town in the eastern part of England. His father, John Boole was a cobbler and his wife Mary, was married for over nine years before George was born. John Boole had a passion for learning and for scientific instruments. Since his family was very poor, he was not sent for a proper formal education. With his father's help, he learned Latin, Greek, French and German. At 19, he opened up his own school in his hometown. Although he was very busy with teaching and managing the operations of, he managed to transform himself from a student of mathematics, to a creative mathematician. During this time, he published a dozen research papers in the Cambridge Mathematical Journal and submitted a very long paper to Philosophical Transactions of the Royal Society.

In 1849,he was appointed Professor of Mathematics at Queen's College in Cork. During his years of research, he was fascinated by Aristotle's logic. He tried to relate his findings to Aristotle's fundamental axiom of all philosophy. Boole died in 1864 at the age of 49.

Based on his finding, we have what is now known as Boolean Algebra. Below is a sample of it [Davis]:

> If x and y represent two classes, then xy represents the class whose members belong to both x and y (Now we call it *intersection*).
>
> Hence the equation $x.x = x$ is always true.
>
> Boole later discovered for any ordinary number, the above is true only for $x=0$ and $x=1$.
> (i.e. $0 . 0 = 0$ and $1 . 1 = 1$)
>
> From there on, Boole interpreted that $0x = 0$ and $1x = x$.

In modern terminology, 0 represents empty set and 1 is the universe of discourse.

Boole later defined the following:

- x+y as being the class of things that can either be found in either class x or class y
- x –y being the class of things that are not in y

If we apply ordinary algebra to Boole's algebra, then

$$xx = x$$
$$\rightarrow \quad xx - x = 0$$
$$\rightarrow \quad x(x-1) = 0$$

The interpretation of the above is that nothing can belong to a class x and at the same time not belong to the class. In today's technology, Boole's findings are widely used in the telephone switching and the design of modern computers. It can also be seen as a fundamental step in today's computer revolution.

## Frege's Begriffsschrift

Gottlob Frege was born on November 8, 1848, in Wismar, East Germany. His father was a theologian and managed a girls' high school, together with his mother. At the age of twenty one, he entered a university at Jena, Two years later he moved to Gottingen University where three years later he obtained his Ph.D. in mathematics. After that, he was appointed as a lecturer, then an Associate Professor at the University of Jena within five years.

Starting from the year 1873, until the end of World War I, Germany was not stable politically, and later on, economically. By the year 1923, Germany faced hyperinflation, which affect most Germans, including Frege. He died in 1925, after he was forced to board with relatives. During this time, he wrote his diary, which was discovered by his adopted son, Alfred.

In 1879, Frege published a booklet entitled "Begriffsschrift" with the subtitle "a formula language. Modeled upon that of arithmetic, for pure thought"[Davis]. This work is what is now known as Predicate Calculus. It is a formal system of two components namely a formal language and logic. The formal language that Frege designed was capable of expressing predication, complex and quantified statements.
Example of Frege's work is as follows:

All cats are mammals
If Tommy is a cat, then Tommy is a mammal

Also,

Some cats are Siamese
Tommy is a cat and it is Siamese

In the present general notation, we would write as:

$\forall$x (if x is a cat then x is a mammal)
($\exists$x) (x is a cat and x is Siamese)

The symbols ∀ (upside down letter A to represent *all*) and ∃ (backwards E) are widely known as the universal quantifier and existential quantifier respectively. He also used the symbols ⊃ to represent "the consequence" (if… then) and the symbol ∧to represent the logical "and" respectively. As part of his predicate calculus, Frege developed a strict definition of a 'proof'. With these findings, Frege was said to have virtually founded the modern discipline of mathematical logic which has now become the standard logic taught to undergraduate in logic courses in mathematics, computer science and philosophy.

## Hilbert's Program for Mathematics

Georg Cantor was born in 1845 in St. Petersburg, Russia to a successful businessman father and a musician mother. In his youth, Cantor became interested in mathematics and he studied under great mathematician at that time, Weistrass, Kummer and Kronecker. After finishing his studies, Cantor was appointed as a lecturer in Halle.

Before this, Frederich Gauss (1777-1855), a German mathematician had stated that infinity should only be used as "a way of speaking' and not as a mathematical value. However, Cantor thought otherwise, that is, he considered infinite sets as a completed entity, not as sets that went on forever. He called these actual infinite numbers transfinite numbers[Vebstas]. His theory on infinite sets reset the foundation of nearly every mathematical field. Cantor's opinion was not well accepted by the mathematicians even though his efforts, with the help of his teacher, Weierstrass and friend Dedekind, were an attempt to meet the demand of conceptual understanding that went beyond symbols. Hilbert described Cantor's work as : " … the finest product of mathematical genius and the supreme achievements of purely intellectual human activity"[Crnkovic]. He made efforts to defend and prove Cantor's transfinite during his presentations in the international congresses of Mathematics in 1900 and 1904.

David Hilbert was born in 1862 in Konigsberg, Prussia. He had remarkable talent in Mathematics and entered University of Konigsberg to study it. During his time, a lot of problems surfaced and conceptual understanding were demanded. In 1888, Hilbert traveled to major centers on mathematics to meet with leading mathematicians. A few things happened in the year 1892 which affected his life: death of Kronecker and retirement of Weiestrass. Resulting from these, the closed world of academic mathematical life in Germany began to unfreeze. After six years of waiting, Hilbert finally was accepted as a regular academician at Konigsberg.

In 1920, Hilbert started a program of metamathematics or proof theory: axioms to be proved consistent were encapsulated in the formal logical system in which a proof is only an arrangement of a finite number of symbols [Davis,Chaitin2]. This program founded the field of "formalism". He came up with this program in order to eliminate the question regarding the foundations of mathematics. It was to be done by turning every mathematical proposition into a formula. Hilbert's program has two parts:

- Provide a single formal system of computation capable of generating all of the true assertions of mathematics from "first principle" (first order logic and elementary set theory)
- Provide mathematically that this system is consistent that is it contains no contradiction.

If successful, all mathematical questions could be established by mechanical computations.

In 1928 Hilbert addressed the International Congress of Mathematicians. He proposed a series of questions. Among the most significant ones are the following three:

- Is Mathematics complete? Can every mathematical statement be proved or disproved?
- Is Mathematics consistent? Is it true that statements such as 1=2 cannot be proven by valid method?
- Is Mathematics decidable? Is there a mechanical approach that can be applied to any mathematical assertion and will tell either that assertion is true or not?

The third one is also known as the Entscheidungsproblem (decision problem). In principle, solving this problem would need an algorithm that would reduced all human deductive reasoning to just calculations. To some extent, it would be an answer to Leibniz's dream. This problem lead to works and development done by later mathematicians like Gödel and Turing.

## Gödel's Theorem

Kurt Gödel was born in Brno, a town in Austro-Hungarian Empire. He went to school in Germany. After secondary school education in Brno, Gödel moved to Vienna in fall 1924. His original intention was to study physics, but he changed to mathematics. The change was due to his fascination on the beauty of patterns of integers, revealed in lectures he attended on theory of numbers.

In 1933, he was offered to go to the Institute for Advanced Study in Princeton. He looked forward to work with Albert Einstein and John Von Neumann. There was no record on his personal life while he was in Princeton. However, manuscripts were collected of his lectures in Cambridge, Massachusetts.

In the course of his studies, he tried to solve Hilbert's problems. In 1931, Gödel answered two of the questions by showing that every sufficiently powerful formal system is either complete or incomplete [Chaitin1, Shalizi]. He also proved that even if an axiom system is consistent, it cannot be proved within itself. He showed that Hilbert's program is impossible. In the process, he discovered that any interpretation of the letters in the formulas with respect to which the premises are true statements, the conclusion is true as well, that is, logic → premise → conclusion. When the symbolic logic of Frege-Russel-Hilbert was used, each premises, each conclusion is represented by a logical formula(strings of symbols). A sample of logical inferences in which the first two lines are premises and the third one is the conclusion, is as follows[Davis]:

Predators have sharp teeth
<u>Wolves prey on sheep</u>
Wolves have sharp teeth

Thus the symbolic inference may be read as

For all x, if there is a y such that x preys on y, then x has sharp teeth.

Note that this set of statements have the same logical features as of Aristotelian Syllogistics, which made Aristotle's ancient logic the starting point of all logics.

## Turing and Neumann

Alan Mathison Turing was born in London on June 23, 1912. His aptitude for the sciences was obvious since the beginning of his school years. He started his career in mathematics in King's College, Cambridge University in 1931 and became a fellow there upon graduation. He then went to Princeton University to for his Ph.D. He was then invited to remain there by Von Neumann who then was on one of the first six professors at the Institute for Advanced Studies.

John von Neumann was born in Budapest, Hungary on $28^{th}$ December 1903. He received his early education in Budapest and entered the University of Budapest in 1921 to study Chemistry. He was a brilliant scholar and published his first paper when he was 18. His areas of interest are set theory, algebra and quantum mechanics. In 1930, he was invited to visit Princeton University, where he met Turing who was pursuing his Ph.D. His interest was more on the use of computers to solve specific problems, unlike his peers who were interested in application to the development of tables. During the war he used his expertise in hydrodynamics, ballistics etc by being a consultant.

It was during this war time that Turing explored the "Turing machine"[Davis, Kowalik]. He described it as a machine that would read series of 1's and 0's from a tape.The Turing machine is hypothetical machine that has an infinite amount of memory [Chaitin 1].

Turing provided a solution Hilbert's Entscheidungsproblem by conceiving a formal idea of a computer. A decade before the first computer was born, Alan Turing created the concepts of what are now known as The Turing Machine and the Universal Turing Machine. The Turing Machine describes a conceptual process of computation and the Universal Turing Machines consists of many Turing machines. Compare these to a set of computer programs and a computer respectively.

## Remarks and Conclusions

This paper intends to re-highlight a well known fact but easily forgotten and overlooked - the roles played by mathematicians in the creation and development of computer have been enormous. Sadly, as computers get more powerful and sophisticated, the more mathematicians "shy away" from computers and the technologies behind them. This phenomenon should not be allowed to go on.

Gone are the days when Computing Science was a branch of Mathematics. Originally known to be parts of Mathematics, Logic, Algorithms and Semantics are now being widely developed as those of Theoritical Computer Science in the from of topics such as Domain Theory, Computational Geometry and Neural networks, to name a few. We strongly believe that mathematics (hence mathematicians) and computers science (or computer technology) should co-exist and to be co-developed to achieve the optimum results and output in research findings (theoretical and applications) as well as in the development of new subject or fields. Encryption is one of such examples where both mathematics and computer technology can be fully utilized. Even in new field like e-commerce, designing a system that is secure is very much a mathematical problem [Schick].

Graph theory and its applications can be widely found in areas such as data-mining, scheduling and logic programming. In short, mathematics is very much alive whether it is known on its own or as part of other subjects such as computer science, engineering or social sciences.

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