# Tangent Construction Made Hard

Jen-chung Chuan
Department of Mathematics
National Tsing Hua University
Hsinchu, Taiwan 300
`jcchuan@math.nthu.edu.tw`

## Introduction

Given a reasonably smooth curve, its tangent can be constructed with ease if either its parametric representation or its polar equation is known. The task becomes hard if we impose the restriction that only synthetic construction be allowed. As illustrated in the paper "No Calculus Please" [1], many routine problems become interesting when solved without the tools of calculus, we wish to demonstrate here how the mundane tangent construction becomes exciting when we focus on the purely synthetic method. In a fully industrialized society the motor vehicles coming out of the assembly lines may not appeal to the consumer as attractive as a hand-made classic model. In the same vein, no two curves to be discussed here share the identical procedure of construction. All constructions can be implemented under the dynamic geometry environment given by Cabri Geometry [2].

## Rose of Three Pedals

In polar coordinates, the equation of the rose of three pedals is given by

$$r = \cos 3\theta$$

The curve is constructed synthetically by taking the locus of the foot of perpendicular of the point with the polar coordinates $(1, 4\theta)$ dropped upon the line through the origin making an angle $\theta$ with the x-axis.
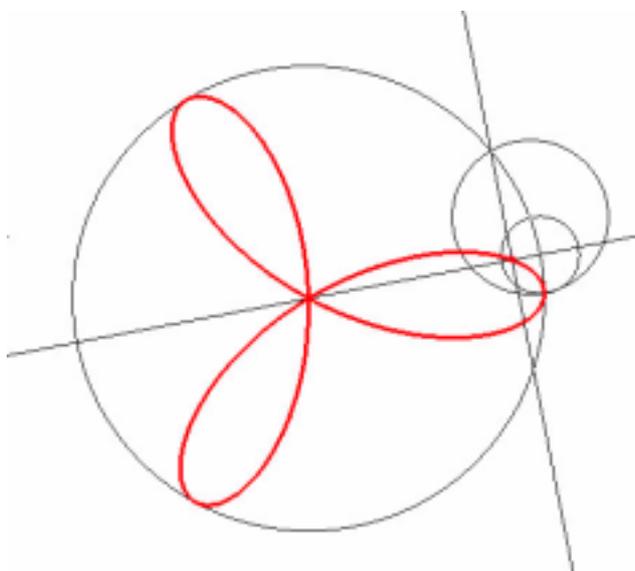


To inspire the construction of the tangent, we need the following sequence of matrix identities:

if

$$\begin{bmatrix} x(\theta) \\ y(\theta) \end{bmatrix} = \begin{bmatrix} r(\theta)\cos(\theta) \\ r(\theta)\sin(\theta) \end{bmatrix}$$

then

$$\begin{aligned}
\begin{bmatrix} x'(\theta) \\ y'(\theta) \end{bmatrix} &= \begin{bmatrix} r'(\theta)\cos(\theta) - r(\theta)\sin(\theta) \\ r'(\theta)\sin(\theta) + r(\theta)\cos(\theta) \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} r'(\theta) \\ r(\theta) \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} r'(\theta) \\ r(\theta) \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r(\theta) \\ -r'(\theta) \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{bmatrix} \begin{bmatrix} r(\theta) \\ -r'(\theta) \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta + \pi/2) & -\sin(\theta + \pi/2) \\ \sin(\theta + \pi/2) & \cos(\theta + \pi/2) \end{bmatrix} \begin{bmatrix} r(\theta) \\ -r'(\theta) \end{bmatrix} \\
&= \begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} r(\theta) \\ -r'(\theta) \end{bmatrix}.
\end{aligned}$$

Hence the tangent must be perpendicular to

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} r(\theta) \\ -r'(\theta) \end{bmatrix}.$$

But this last point is easily located: from the point $P$ on the curve search for a point $Q$ along the perpendicular to $OP$ at $P$ at a distance of $-r'(\theta)$ away from $P$. In this case,

$$-r'(\theta) = 3\sin(3\theta)$$

so the normal direction is obtained by the straight line $OQ$; the perpendicular to $OQ$ drawn through $P$, therefore, is the required tangent:
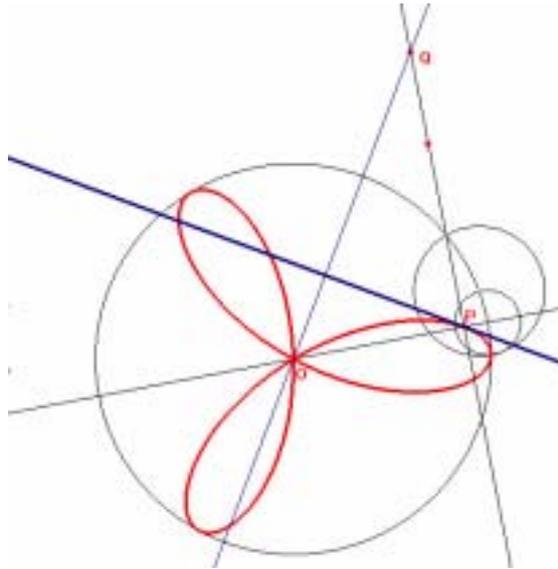
# Double Folium

In polar coordinates, the equation of the double folium takes the form

$$r = \sin\theta\,\sin 2\theta.$$

The curve is constructed synthetically by taking the locus of the foot of perpendicular through $A(0, \sin 2\theta)$ dropped upon the line passing through the origin making an angle $\theta$ with the x-axis:
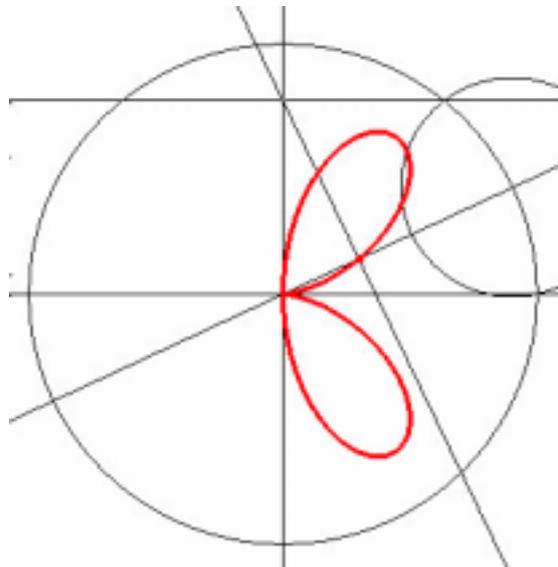
To construct the tangent, we first compute

$$r' = \cos\theta\sin 2\theta + 2\sin\theta\cos 2\theta.$$

Since $AP = \cos\theta\sin 2\theta$, we are to locate $Q$ on $AP$ with $PQ = -r'$. To this end, the line $L$ parallel to $OP$ is drawn through $B(\cos 2\theta, 0)$, and then the point $Q$ is obtained by reflecting point $A$ with respect to $L$. The construction of the required tangent is completed by drawing the line through $P$ perpendicular to the line $OQ$.
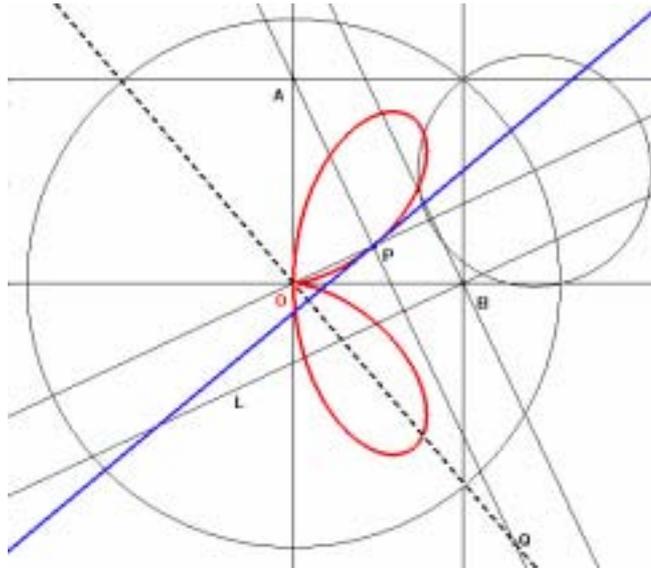
# Lemniscate of Gerono

Lemniscate of Gerono has the parametric representation

$$x = \cos t$$

$$y = \sin t \cos t$$

$$0 \leq t \leq 2\pi.$$

The curve is constructed synthetically by taking the locus of the foot of perpendicular $P$ through $(\cos\theta, \sin\theta)$ dropped upon the line half way between $A(\cos 2\theta, \sin 2\theta)$ and the x-axis:
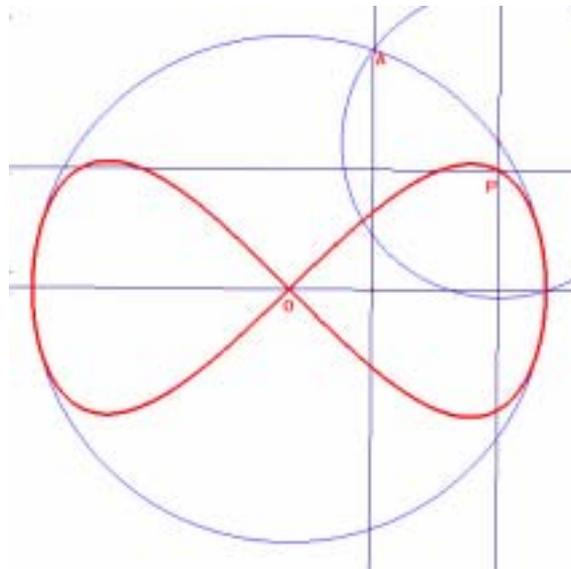


To construct the tangent, we first compute

$$x' = -\sin\theta$$

$$y' = \cos 2\theta$$

The tangent is therefore perpendicular to $OQ$ with $Q(\cos 2\theta, \sin\theta)$.
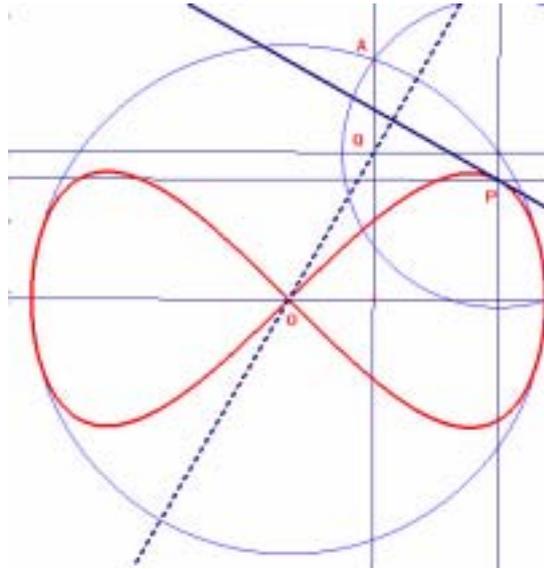
# Freeth's Nephroid

In polar coordinates, the equation of the Freeth's Nephroid is given by

$$r = 1 + 2\sin\frac{\theta}{2}.$$

The curve is constructed synthetically by taking the locus of the reflection of the point $(-\cos\theta, -\sin\theta)$ with respect to the perpendicular drawn from $(\cos\frac{\theta+\pi}{2}, \sin\frac{\theta+\pi}{2})$ to the line through the origin making an angle $\theta$ with the x-axis.

In order to construct the tangent, we proceed to compute $r'$ :

$$r' = \cos\frac{\theta}{2}.$$

After locating $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} r(\theta) \\ -r'(\theta) \end{bmatrix}$, the construction becomes straightforward.

# Folium of Descartes

Despite its simplicity, the equation

$$x^3 + y^3 = 3xy$$

does not help constructing the curve synthetically. Turning to polar coordinates, we obtain, after the substituting $x = r\cos t, y = r\sin t$,

$$\frac{3}{r} = \cot t \cos t + \sin t \tan t.$$

The curve can be constructed by taking the inverse of the point with polar coordinates

$$(\frac{\cot t \cos t + \sin t \tan t}{3}, t)$$

with respect to the unit circle.
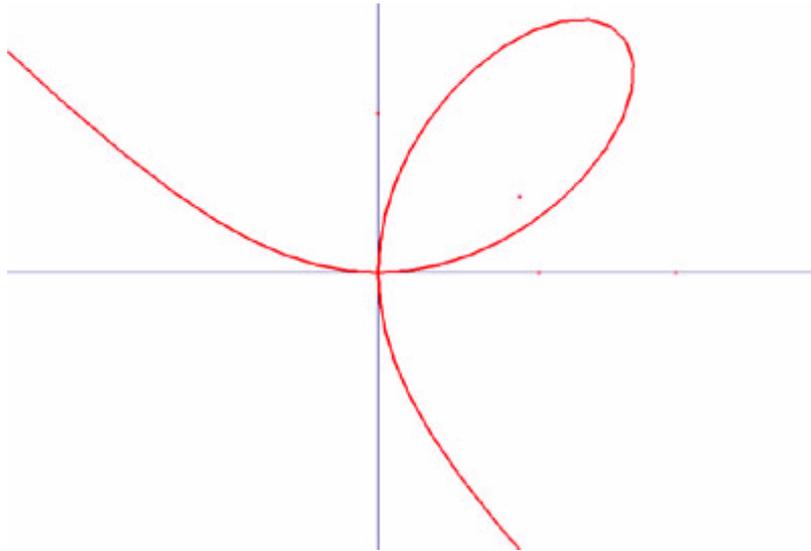
The tangent construction becomes difficult if we are guided by the derivative $r'$ as in the previous constructions. At this point the formula

$$x^3 + y^3 = 3xy$$

all of a sudden becomes illuminating: by taking derivatives we have

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x};$$

so the slope of the normal appears as

$$-\frac{dx}{dy} = \frac{y^2 - x}{x^2 - y}.$$

In plain words: the tangent is a straight line passing through the point $(x, y)$ and perpendicular to the straight line determined by the points $(x^2, y^2)$ and $(y, x)$.
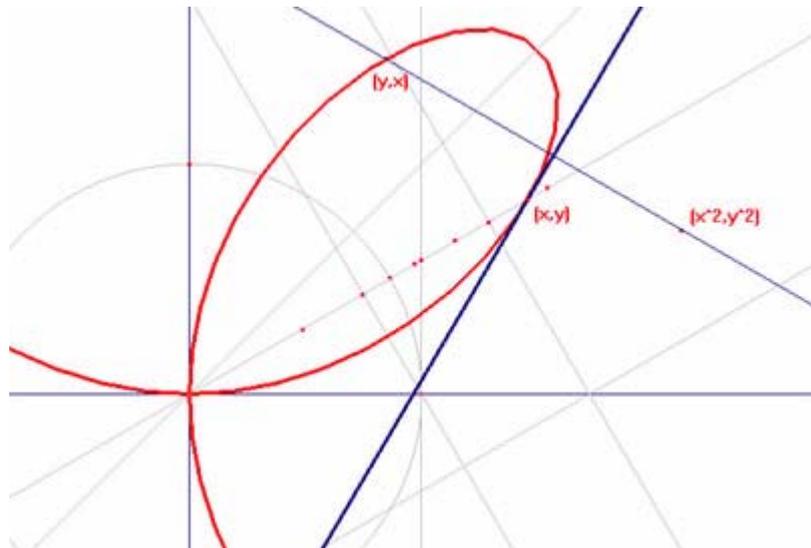


# Conclusion

The synthetic construction of the tangent described here may not be the easiest. It does, however, make student to concentrate on the procedure. Training people to think, after all, is what math education is about.

The CabriJava applets associated with this paper can be found in:
http://steiner.math.nthu.edu.tw/ne01/tjy/dynamic/contants/FamousCurves-e.html

# References

1. J.H. Bitchart and Leo Moser, No Calculus, Please, Script Math., Vol. 18 (1952),221-236.

2. Cabri Geometry, http://www.cabri.net/index-e.html

3. The Purple Bear's Web, http://purplebearsweb.co.uk/Fly/Knots/Why_collections_of_small_thing/why_collections_of_small_thing.html

4. E.H. Lockwood, A Book of Curves, Cambridge U. Press, 1963.