The engineer's new pocketknife: the graphics calculator

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Abstract

The modern pocketknife, with its many features from corkscrew to can opener, is often viewed as an essential tool to carry at all times and to be used when needed. Students and teachers of engineering mathematics can view the graphics calculator in the very same way. It can be used at the tertiary level to considerably enrich the student's understanding and enjoyment of many of the topics covered in engineering mathematics units. The calculator is not a replacement for mathematical understanding, and indeed it requires some level of mathematical understanding for its effective use. However, with its combination of numerical, symbolic and graphical routines, it can take the time and tedium out of the calculations and at the same time provide the graphs which enhance the understanding of the concepts and help to provide a fuller picture of the topic being studied. This is the approach taken at our university. In this paper we elaborate on this theme, and we provide relevant illustrations of the use of modern graphics calculator technology in the engineering mathematics classroom.

1   Introduction

From the point of view of both breadth and depth, the description 'Engineering Mathematics' encompasses a wide range of mathematical topics. It includes elementary calculus and linear algebra, the study of differential equations and vector calculus, Fourier theory and the study of signals and systems. All of these include a great deal of necessary computation. Previously this calculation was extremely time-consuming and occupied the majority of the student's time. Quite often the mathematical forest was lost in the computational trees. This undesirable state of affairs has been ameliorated by the advent of the graphics calculator. This is particularly true of the new breed of graphics calculators, exemplified by Hewlett Packard's HP 49G model, which incorporates a computer algebra system (CAS).

We view the graphics calculator as a tool to facilitate teaching and learning in engineering mathematics. It should not be thought of as a replacement for mathematical understanding. In fact, a real understanding of the mathematics involved is often essential for the optimal use of the calculator to carry out the relevant computation or to obtain the desired graphical display. In the variety of uses and the technical toolbox contained in the graphics calculator we see an analogy in some way with the modern pocketknife. This item, with its many features from corkscrew to can opener, is often viewed as an essential tool to carry at all times and to be used when needed. Students and teachers of engineering mathematics can view the graphics calculator in the very same

way. This is precisely the approach we have taken in our mathematics units at Edith Cowan University (ECU). We impress on our students that the available calculator and computer technology can be of enormous assistance to them but its help is to a large extent bounded by the mathematical expertise of the user.

Graphics calculators have been permitted in the Western Australian Tertiary Entrance Examination since 1998. (An analysis of their effect on student performance in the TEE Calculus examination can be found in [6].). At ECU we make the assumption that each student in our mathematics units has access to his or her own graphics calculator and has basic knowledge of its use. Students in first year calculus units have been able to make use of our workshop notes on the HP38G and the HP48G graphics calculators (see [1] and [2]). However we have not attempted to cover all brands and have not run special calculator workshops for these students. Graphics calculator use (and education) has been incorporated into the regular class sessions and formal assessment (see [5]). To date, the vast majority of students had graphics calculators but not CAS graphics calculators. The few exceptions to this were students who bought a new calculator late this year. However these students turned out not to be unduly advantaged because they were either unable to learn how to use these sophisticated tools in the given time frame or they were unaware of the power quantum leap of these calculators.

However, the use of the CAS graphics calculator in teaching and assessment will impact on both the way in which Engineering Mathematics units will be taught and how and what is assessed. This is the challenge we are coming to grips with here at ECU for the coming year. To illustrate the need for such a rethink and rewrite, we give in this paper some examples of the use of modern CAS graphics calculator technology in the Engineering Mathematics classroom. These examples should not in any way be regarded as an exhaustive set and do not include topics that have been well handled by ordinary graphics calculators for some time now and which can be found in the literature (see, for example, [3] and [7]). The capabilities of the CAS graphics calculator will be illustrated here via the HP 49G graphics calculator.

2. Limits

Evaluation of limits is a task that students have often found very difficult. This is particularly true in the case, such as engineering, where the student is studying mathematics as a service unit. This is an area where using a CAS graphics calculator makes a real difference, particularly in the assessment. The following example is from one of our first year calculus examinations.

Example 2.1: Evaluate the following limits:

$$\text{(a)} \quad \lim_{x \to 1} \frac{\tan(x-1)}{x^2 - 1} \qquad \text{(b)} \quad \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$$

Traditionally the student would have been expected to apply algebraic methods or L'Hopital's rule to evaluate the indicated limits. With the advent of the scientific calculator the student could look at a table of values to conjecture that the answers are 0.5 and e respectively. With the advent of the graphics calculator the student could look at the respective graphs to suggest the values, although it should be noted that using a graphics calculator to graph the function in (a) is a serious task in its own right. However this example is reduced to a trivial exercise with the use of a CAS graphics

calculator and no longer warrants consideration as a major exam question in its own right. On the HP49G one works in *Exact* mode in the Equation Writer (*EQW*) and then uses the LIMIT function from the LIMITS & SERIES submenu in the Calculus (*CALC*) menu to evaluate each limit directly. The answers shown in Figure 1 actually represent a composite view of the input limit and the output value, making use of the editing property of the HP49G calculator.
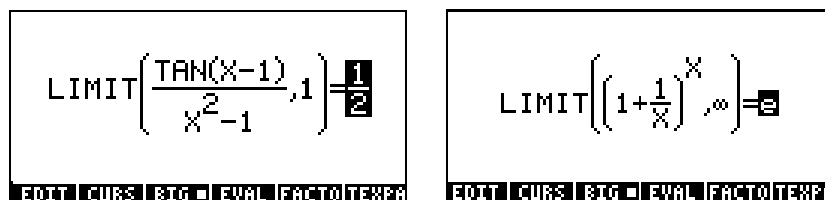

Figure 1: Limits

3 Integration

3.1 Definite Integrals

Evaluating a definite integral on a graphics calculator is straightforward. With a CAS graphics calculator one can, where it is relevant, obtain an exact value rather than just an approximation. The use of the CAS graphics calculator enables students to handle applications that involve definite integrals for which they have no analytical evaluation technique.

The following examples are from our first year Elementary Calculus tests. The first is a standard example, although these students do not have the familiarity with the trigonometric identity that is used in its analytic evaluation, while the second integral is one for which the students would not have any available analytic technique.

Example 3.1.1:   Evaluate the following integrals:

(a) $\displaystyle\int_0^{2\pi} \sin^2(x)\, dx$ (b) $\displaystyle\int_{0.5}^1 \frac{1}{x^2\sqrt{1+x^2}}\, dx$

On the HP 49G the student can again work in exact mode in the Equation Writer and simply evaluate (*EVAL*) the integral written in the standard form. The results, again in composite screen format, are given in Figure 2.
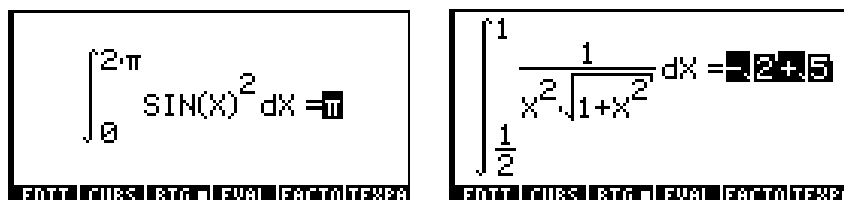

Figure 2: Definite single integrals

The HP49G calculator can also be used to evaluate multiple integrals. This can be done stepwise as well as directly which can be useful for the student in understanding what is involved in the evaluation of a multiple integral. This process is illustrated in Example 3.1.2 for the evaluation of the double integral arising from the calculation of the area enclosed by the cardioid given by the equation $r = 1 + \cos\theta$.

Example 3.1.2:   Evaluate stepwise the double integral $\int_0^{2\pi} \int_0^{1+\cos\theta} r \, dr \, d\theta$.

On the HP49G the student should first enter the double integral in the Equation Writer in standard form. The student should then select the inner integral and evaluate it, then select the outer integral and again evaluate. This process is shown in Figure 3.
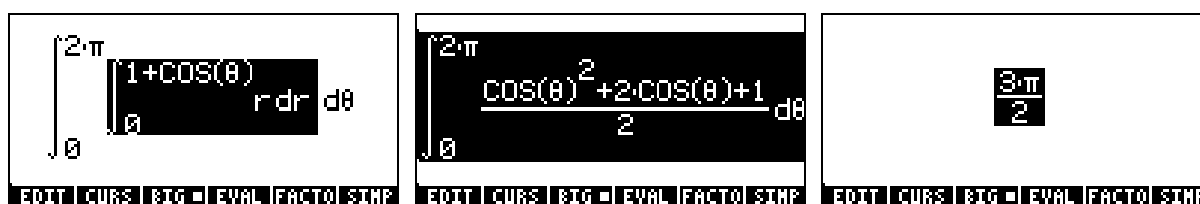
Figure 3: Definite double integral

3.2   Indefinite Integrals

The incorporation of a CAS has boosted the power of the graphics calculator in this regard. Previously the graphics calculator provided no advance over a simple table of antiderivatives. For instance, good as it is in many ways, the HP48G graphics calculator cannot handle the indefinite integral evaluated in Example 3.2.1 by the CAS property of the HP49G calculator.

Example 3.2.1:   Evaluate $\int e^{4x} \sin x \, dx$.

To find an antiderivative of $e^{4x} \sin x$ on the HP49G the student should enter the Equation Writer and then use the INTVX function from the DERIV. & INTEG. submenu in the Calculus menu (making sure the complex mode is not on). The result is shown in Figure 4.

Figure 4: Indefinite integral (parts)

Example 3.2.1:   Evaluate $\int \frac{x}{x^3 - 1} \, dx$.

As for Example 3.2.1, an antiderivative can be found directly. In this case the problem and the solution (over two screens to see it all) are given in Figure 5.
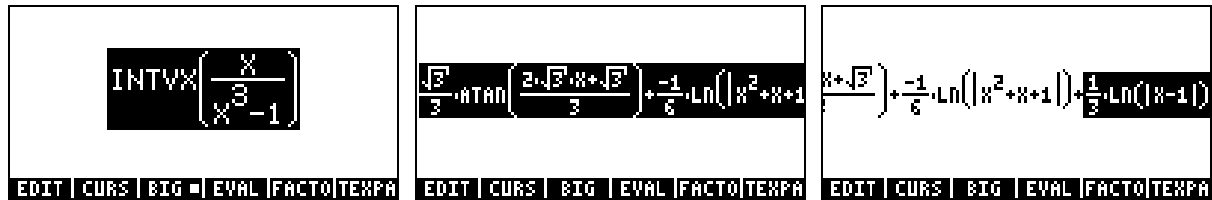

Figure 5: Indefinite integral (partial fractions)

To detail the process for the student, the evaluation, using the method of partial fractions, can be done stepwise. To carry out the partial fractions expansion on the HP49G the student should use the PARTFRAC command from the POLYNOMIAL submenu in the Arithmetic (*ARITH*) menu. The result is shown in Figure 6.
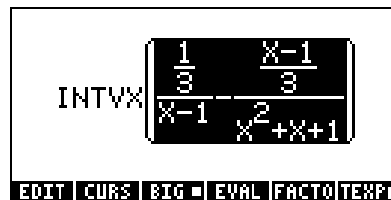
Figure 6: Partial fractions

The student can then rewrite the original integral as the sum of the resulting two integrals and evaluate each of these separately to obtain the final solution. The details are shown in Figure 7.
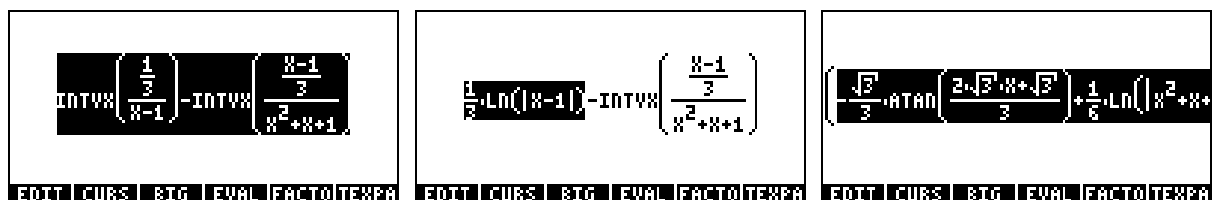

Figure 7: Stepwise integration

3.3   Improper Integrals

Engineering Mathematics students are often faced with the need to evaluate an improper integral. Involving as it does the evaluation of a limit as well as an integral, the CAS graphics calculator is doubly useful in such cases. The following example is from our first year calculus unit.

Example 3.3.1:   Evaluate $\int_1^\infty \frac{1}{x(x+1)}\,dx$

In contrast to the rather complicated procedure needed for the (non CAS) HP48G graphics calculator, this improper integral can be easily evaluated by making use of the CAS on the HP49G graphics calculator. The student simply enters the integral in standard form on the Equation Writer and evaluates. The result, making use of the editing properties of the HP49G, is given in Figure 8.

$$\int_1^\infty \frac{1}{X\cdot(X+1)}\,dX = \boxed{\text{LN(2)}}$$

EDIT | CURS | BIG ■| EVAL |FACTO|TEXPA

Figure 8: Improper integral

3.4   Integral Transforms

Although a CAS graphics calculator can easily be used to evaluate improper integrals, there are some special improper integrals that arise frequently in the study of Engineering Mathematics. The most common ones are the Fourier and the Laplace integral transforms. The HP49G provides a special command for these.

Example 3.4.1:   Evaluate the Laplace Transform of $e^{4x}\cos x$

On the HP49G one works in *Exact* mode in the Equation Writer and then uses the LAP function from the DIFFERENTIAL EQUATIONS submenu in the Calculus (*CALC*) menu to find the required Laplace Transform. The problem and the answer are shown in Figure 9.

LAP(EXP(4·X)·COS(X))

EDIT | CURS | BIG ■| EVAL |FACTO|TEXPA

$$\frac{X-4}{X^2-8X+17}$$

EDIT | CURS | BIG ■| EVAL |FACTO|TEXPA

Figure 9: Laplace transform

The HP49G calculator can also be used to find inverse Laplace transforms. Again one works in *Exact* mode in the Equation Writer and then uses the ILAP function from the DIFFERENTIAL EQUATIONS submenu in the Calculus (*CALC*) menu to find the required Inverse Laplace Transform. The problem and the answer are shown in Figure 10.

Example 3.4.2:   Find the Inverse Laplace Transform of $\dfrac{s-1}{s^2(s+1)}$.
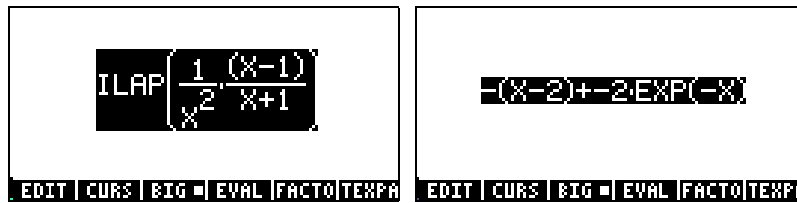
Figure 10: Inverse Laplace transform

To check this result on the HP49G the student could first rewrite the given expression as $\dfrac{x-1}{x^2(x+1)} = \dfrac{1}{x^2} - \dfrac{2}{x^2(x+1)}$ and then apply the Inverse Laplace Transform to each term. The process and the final result are shown in Figure 11.

Figure 11: Stepwise Laplace transform

4   Orthogonal Polynomials

Systems of Orthogonal Polynomials also arise in Engineering Mathematics. The most commonly used orthogonal polynomials can be calculated directly on the HP49G calculator. This is illustrated here for the Legendre polynomials.

Example 4.1: (a)   Evaluate the Legendre Polynomial of degree 4.
                (b)   Use the Legendre Polynomials of degree 2 and degree 3 to illustrate the orthogonality property.
                (c)   Evaluate the norm squared of the Legendre Polynomial of degree 4.

To answer these on the HP49G calculator the student works in the Equation Writer and uses the LEGENDRE function in the POLYNOMIAL submenu of the Arithmetic menu. The results are shown (from left to right respectively) in Figure 12.

Figure 12: Legendre polynomials

5 Series Approximation

In this section we shall consider the representation of a suitable function by an infinite series and its approximation by a partial sum of such series. We detail Taylor Series, Legendre Polynomial Series expansion and Fourier Series expansion.

5.1 Taylor Polynomial Approximation

The HP49G has a general Taylor Series Taylor Polynomial capability. It also has two special functions to evaluate a Taylor Polynomial about 0. These are TAYLR and TAYLOR0. The former is also available in the (non CAS) HP48G and has probably been left for upward user compatibility. The latter is restricted to calculating Taylor Polynomials of degree 4. We illustrate both the general Taylor series, together with TAYLOR0.

Example 5.1.1: Find the Taylor Polynomial of degree 4 of $\sin(2x)$ about $x = 0$.

To do this on the HP49G the student uses the Equation Writer, together with the TAYLOR0 function from the LIMITS & SERIES submenu of the Calculus menu. The problem and its solution are shown in Figure 13.
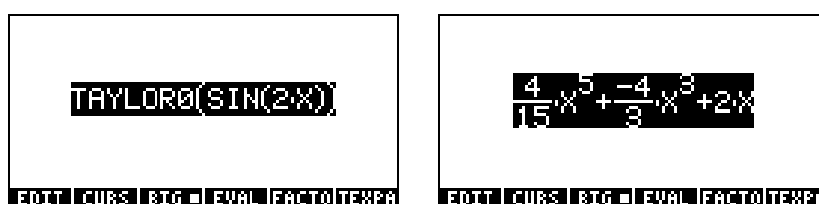

Figure 13: Taylor polynomial of order 4

Example 5.1.2: (a) Calculate the Taylor Series expansion of order 6 for $\ln x$ about $x = 0$
  (b) Illustrate graphically the use of the Taylor polynomials about $x = 0$ of degree 6 and degree 10 to approximate $\ln x$.

On the HP49G the student can obtain the relevant Taylor Series expansion by using the SERIES command in the LIMITS & SERIES submenu of the Calculus menu. The HP49G supplies the expansion in terms of variable h together with the 6th order remainder and the information that, in this case, $h = x - 2$. The syntax and output can be seen in Figure 14.
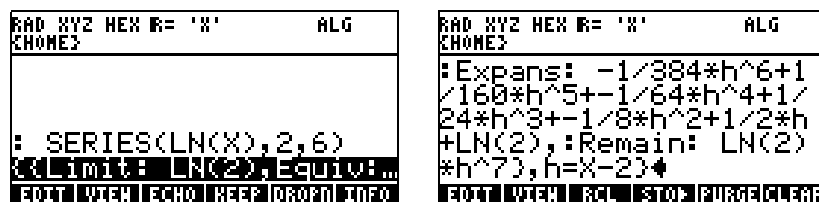

Figure 14: Taylor series for $\ln x$ about $x = 2$

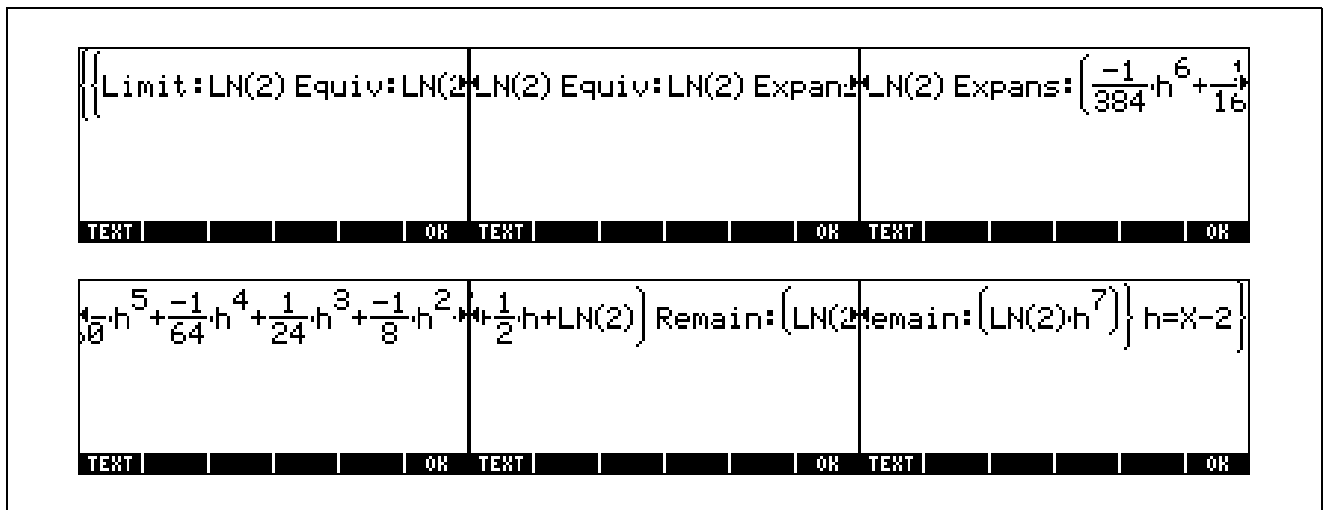Viewing the expansion and scrolling through gives the display indicated in Figure 15.

Figure 15: Taylor series expansion

It is possible then to appropriately edit this series expansion and substitute $x - 2$ for h. This is shown in Figure 16.
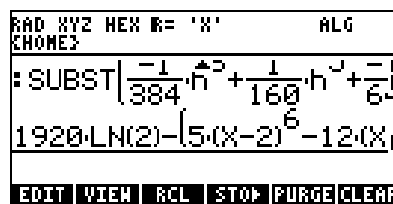

Figure 16: Taylor series in terms of (x-2)

The graph of $\ln x$ together with the approximations about $x = 2$ by the Taylor polynomials of order 6 and 10, separately and together are shown in Figure 17.
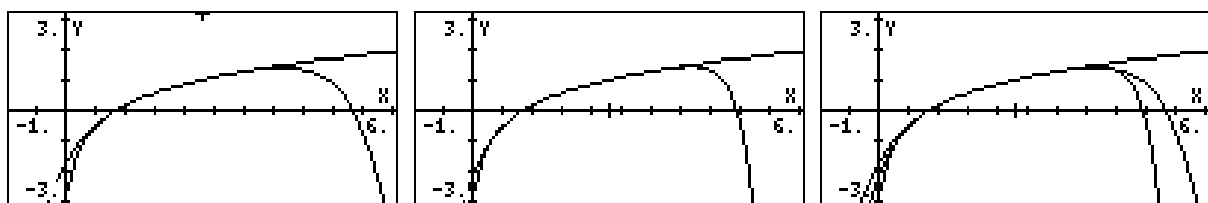

Figure 17: Graphs of Taylor approximations

These graphs can be used to conjecture that the interval of convergence of this Taylor Series representation of $\ln x$ is (0, 4). Note that Taylor Polynomials give a good local approximation to a function.

5.2   Orthogonal Polynomial Approximation

This is illustrated here by means of Legendre Polynomials.

Example 5.2.1:   Approximate $f(x) = x^4$ by the quadratic polynomial of best fit on the interval $[-1, 1]$.

The required approximation can be calculated using the first three Legendre polynomials. On the HP498G the resulting expression can be entered directly into the Equation writer. This is shown in Figure 18.
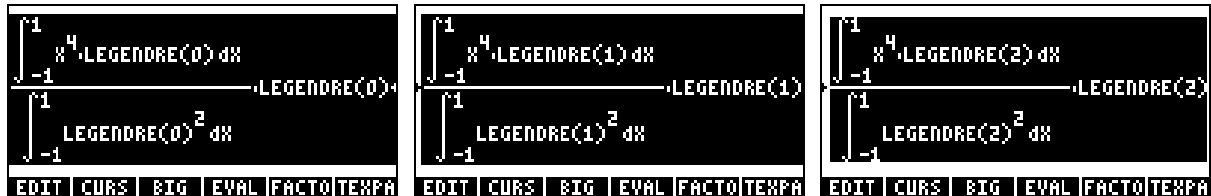

Figure 18: Approximation of $f(x) = x^4$ by the polynomial of best fit on $[-1, 1]$.

The integral expressions and the Legendre Polynomial in each term can then be evaluated and simplified. The results are shown in Figure 19.
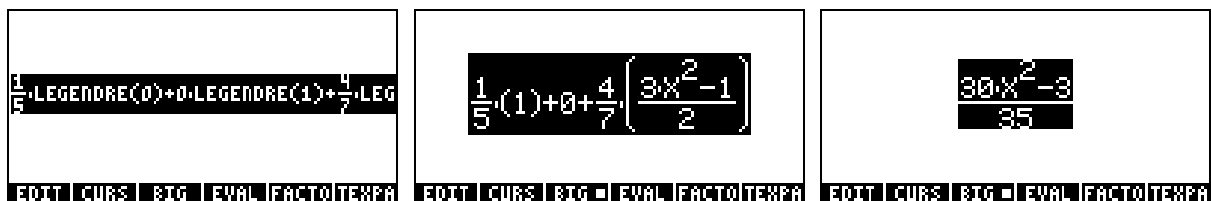

Figure 19: Legendre polynomial approximation

The graphing capability of the HP49G can then be used to display the graphs of x^4 and the desired Legendre Polynomial approximation over the interval [-1,1]. This is shown in Figure 20.
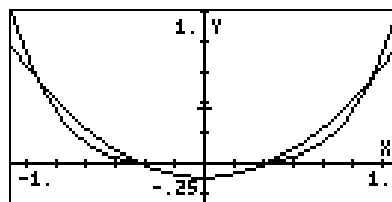

Figure 20: graphs of Legendre polynomial approximation

5.3   Fourier Series Approximation

In this case we approximate a function by a trigonometric series. Trigonometric polynomials are particularly useful for approximating periodic functions. Such functions arise in many engineering applications. Examples include sound waves and electrical current. Trigonometric polynomials give a good global approximation to a function.

Example 5.3.1:   Find the fifth partial sum of the Fourier series of $f(x) = x$ on $[0, 2\pi]$.

To do this on the HP49G is rather more involved. The FOURIER function from the DERIV & INTEG submenu returns complex Fourier coefficients. This can be combined with the RE and IM functions from the Complex (*CMPLX*) menu to define the real Fourier coefficients. These can then be used to define as a function the fifth Fourier partial sum. This is illustrated in Figure 21.
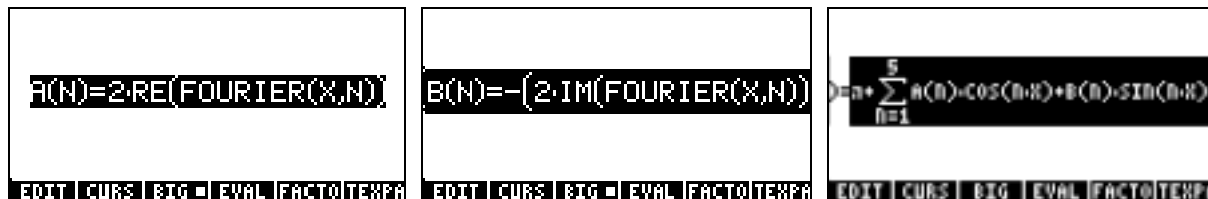

Figure 21: Fourier series approximation

The graphs of the function and the partial sum both separately and together are shown in Figure 22.
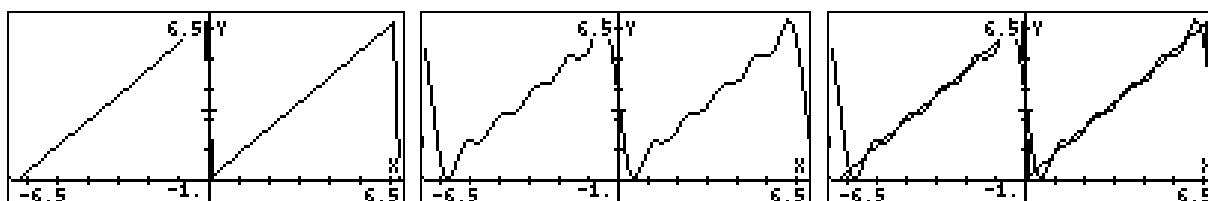

Figure 22: Graph of $f(x) = x$ and the fifth partial sum of its Fourier series on $[0, 2\pi]$.

6. Differential Equations

Several units in Engineering Mathematics deal with solutions of differential equations. We shall illustrate here the ability of a CAS graphics calculator to solve directly two (one first order linear and the other second order linear with constant coefficients) ordinary differential equations of the type standard on first or second year calculus examination papers.

Example 6.1.1: Solve the first order differential equation

$$\frac{dy}{dx} + \frac{y}{x} = e^{-x^2}$$

On the HP49G the student can use the DESOLVE command from the DIFFERENTIAL EQNS submenu of the Calculus menu, (or directly from SSLV on the keyboard) as shown in Figure 23, to obtain the required solution.
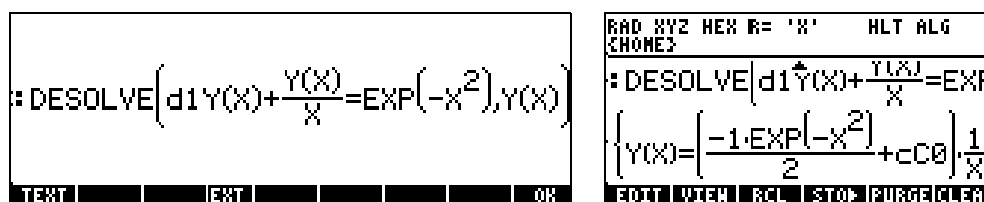

Figure 23: First order linear ODE

Example 6.1.2: Solve the second order differential equation

$$y'' - 4y' - 5y = 5x$$

To solve this differential equation on the HP49G the student uses the LDEC command from the DIFFERENTIAL EQNS submenu of the Calculus menu, (or directly from SSLV on the keyboard). The input for the problem consists of the function on the right hand side of the given differential equation and the characteristic polynomial of the corresponding homogeneous differential equation as shown in Figure 24.
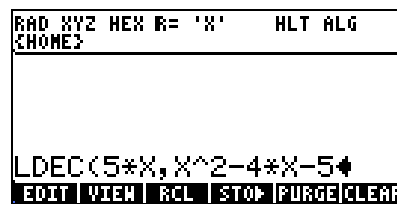
Figure 24: Input for solution of second Order Linear ODE with constant coefficients

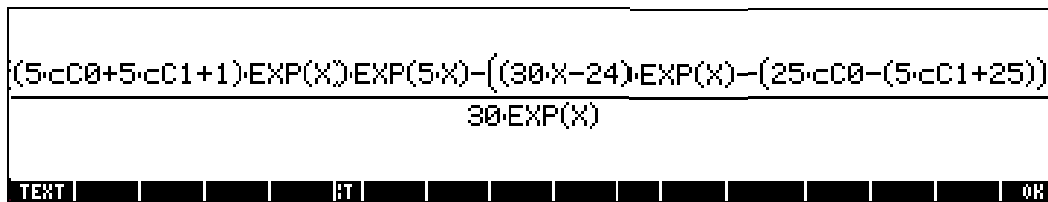The solution of the differential equation is shown in Figure 25.

Figure 25: Solution of second Order Linear ODE with constant coefficients

Although this solution is more complicated, it is clear that it is equivalent to the solution, namely $y(x) = ae^{5x} + be^{-x} - x + 4/5$, that the student would obtain by hand. See [8] for a more detailed investigation of how the HP 49G can be used in the study of differential equations.

6.   Vector Calculus

The CAS graphics calculator can be used to calculate directly functions such as *Div* and *Curl* that arise in vector calculus problems.

Example 6.1: A fluid has velocity field given by

$$\mathbf{v}(x, y, z) = xy^2 z^2 \mathbf{i} + x^2 yz^2 \mathbf{j} + x^2 y^2 z \mathbf{k}$$

(a)   Calculate $\nabla \cdot \mathbf{v}$ and use this to decide if the flow expanding, contracting or incompressible?

(b)   Calculate $\nabla \times \mathbf{v}$ and use this to decide if the flow is rotating or irrotational?
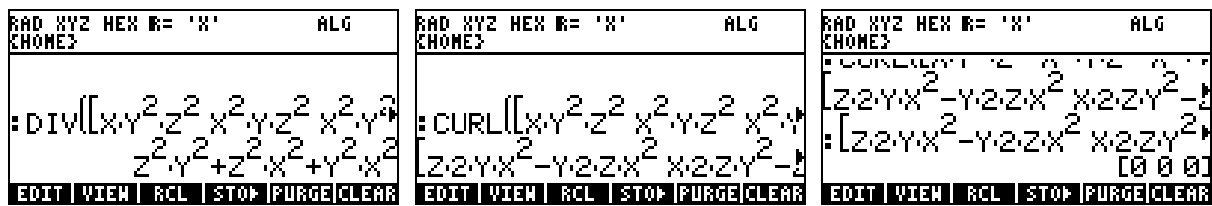
Figure 26: Div (left) and Curl (centre and right)

Here the student would note that the divergence of the given vector field is a sum of squares so that the corresponding flow is expanding and from the fact that the curl is **0**, that the flow is irrotational.

7. Implications for Curriculum and Assessment

Even from the small selection of examples we have been able to give in this paper, it is abundantly clear that student use of a CAS graphics calculator in learning and assessment will have a profound effect on the Engineering Mathematics classroom and subsequent assessment. As opposed to units directed to mathematics students who will hopefully go on to prove new theorems and explore the mathematical frontiers, the students in Engineering Mathematics units have no such ambitions. The objective of the teacher is to ensure that the student has a real understanding of the mathematical concepts covered and the techniques necessary to apply these, together with a sound basis for independent learning of new mathematical concepts as they arise in their future careers. It does not detract from the solution of the problem if these techniques are more closely connected with a the use of a CAS rather than the old pen and paper solutions. What is important is that the student has enough understanding of the mathematics to actually use a CAS graphics calculator sensibly. On a particular occasion the calculator, like any other technology, may give an incorrect answer for any number of reasons, some relating to the user and others relating to the architecture of the particular calculator chosen. See [5] for examples of these arising from the use of the CAS package *Scientific Notebook*. What matters is that the student knows what is reasonable with regard to an answer.

What will having access to a CAS graphics calculator mean? On the one hand it eliminates the need for the use of expensive computer laboratories to have access to CAS – so the argument of undue expense which at times mitigated against the use of CAS in larger mathematics classes will hopefully lose its force. On the other hand, designing meaningful learning activities that allow the student to explore mathematics using the technology will be of even greater importance. This is particularly so given the complexity of these calculators, and this will provide a real challenge for the teachers of these subjects. This will undoubtedly mean that more class-time needs to be spent on such activities than is currently done in laboratories.

Portability also negates the excuse of lack of access to the technology – a favourite one for those students who are either afraid of mathematics or of technology or both. The calculators are sufficiently portable to be carried into examinations and as the examples show they appear to invalidate or devalue the usual kind of questions posed in Engineering Mathematics examinations, provided that students have sufficient working knowledge of the CAS graphics calculator. But looking on the bright side of things, because the calculators are portable students will much more readily expect them to be used in examinations and so might take this technology much more seriously as a result. At least in the case when the student fully understands how to use the CAS to his/her advantage, he/she should be able to demonstrate mathematical understanding much more

easily and without the pitfalls of poor algebra skills for example. In our view this a level of understanding beyond that of simply applying recipes, and applies in particular to the critical assessment of the computational results. In this lies a danger that the gap between mathematically able user and the user of lower ability will become ever wider.

7.   Conclusion

In this paper we have discussed the important question of the impact of the easy availability of CAS graphics calculators on the teaching, learning and assessment of Engineering Mathematics. We have given some examples to illustrate the properties of these calculators and their potential to radically change the Engineering Mathematics classroom. What was a complicated examination paper a short time ago is now a relatively easy task for the student who has an understanding of the mathematics involved and who is a competent user of the CAS graphics calculator. This does not necessarily mean that the examination papers should become harder or that many more topics should be added to the curriculum. However, the best way to incorporate CAS graphics calculator use into the teaching and learning of Engineering Mathematics poses a real challenge and one which needs to be met with careful thought as well as with enthusiasm.

8.   References

[1]   Anderson, M., Bloom, L, Mueller, U. and Pedler, P., (2000), HP 38G Graphics Calculator: Notes for calculus, Edith Cowan University, Perth.

[2]   Anderson, M., Bloom, L, Mueller, U. and Pedler, P., (2000), HP 48G Graphics Calculator: Notes for calculus, Edith Cowan University, Perth.

[3]   Anderson, M., Bloom, L, Mueller, U. and Pedler, P. (1999), *The impact of the graphics calculator on assessment and modelling*, The International Journal of Mathematical Education in Science and Technology, **30**, 489-498.

[4]   Bloom, L. and Bloom, W., (2000), *Treat technology with care: Some traps for the unwary*, Proceedings of the Fifth Asian Technology Conference in Mathematics (to appear*).*

[5]   Bloom, L. and Mueller, U., (2000), *Modern Elementary Calculus: Assessment with technology*, Proceedings of the Fifth Asian Technology Conference in Mathematics (to appear*).*

[6]   Forster, P. and Mueller, U. (2000), *Trends in assessment in calculus on the introduction of graphics calculators*, Proceedings of the Fifth Asian Technology Conference in Mathematics (to appear*).*

[7]   Kissane, B. (1995). *Exploring calculus with graphics calculators.* Australian Senior Mathematics Journal, **9**, 7-16.

[8]   Springer, G. T. (1999), Proceedings of the Fourth Asian Technology Conference in Mathematics (eds W-C. Yang, D. Wang, S-C.Chu, G. Fitz-Gerald),