USING TECHNOLOGY TO IMPROVE THE QUALITY OF STUDENTS' WRITTEN MATHEMATICS

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Abstract

In the continuing discussion on the use of technology in mathematics education, a position sometimes taken, with plausibility, is that when students are capable of solving problems without technology, there is no need to introduce it. The authors believe that even the most capable students in the most traditional curriculum can enrich their experiences and improve the quality of their work by using appropriate technology well in a supporting role, and for less capable students technology provides real opportunities to increase understanding and confidence. To assist in directing such use of technology, the authors have developed a flexible five-stage process which in the most detailed problems leads students through initial exploration to multi-stranded investigation of the problem given: the process is illustrated in detail in an example.

Introduction

In 2002, non-symbolic graphics calculators will be allowed to be used by students in the GCE Advanced-level Further Mathematics (university entrance) examination in Singapore for the first time. The Singapore-Cambridge General Certificate of Education Advanced Level Examination is a national examination conducted in Singapore annually. It comprises two three-hour papers. The first covers topics from algebra, analysis, complex numbers and matrices and vector spaces. The second paper covers topics from mechanics and inferential statistics (for syllabus details, see the Singapore Ministry of Education web site, http://www1.moe.edu.sg/). Clearly, there is an advantage to students in having a graphics calculator when attempting the statistics section of the paper, as it automates the long and tedious computations that must be performed when applying the techniques of statistical inference, and that is readily accepted by teachers. However, because of the highly symbolic nature of the remaining content, many teachers are of the view that the value to students of having access to a non-symbolic graphics calculator in the examination is somewhat limited. In essence the teachers are saying that, if the technology cannot directly replace the need to perform the task, or offers only an alternative method of arriving at the required answer, then the calculator offers little value to the user. The authors challenge this one-dimensional view of the role of technology in mathematical activity. Even when the development of pencil-and-paper mathematical skills is a goal, the authors show that much can be done with a graphics calculator to support these skills and to improve the quality of student responses. They leave aside the issue of the need for developing some of these pencil-and-paper manipulative skills to such a high level of competence, when graphics calculators

with symbolic processing capabilities are waiting in the wings (see for example, Demana and Waits, 1998; Kissane, 1999; Ralston, 1999).

In this paper, we show how students can use a technology such as a non-symbolic graphics calculator as a supporting medium to improve the quality of their pencil-and-paper work. To this end a flexible five-stage process, acronymically termed ESCAPE, is proposed. In the most extended and demanding problems, of the sort that characterize advanced-level mathematics courses, the ESCAPE process leads students through initial exploration to multi-stranded exploration of the problem to be solved. The process is illustrated using an examination question taken from a typical A-level mathematics examination paper, and is offered both because of its potential to broaden the view of the role of technology in mathematical activity, but also to encourage good mathematical practice, which should be a goal of any mathematical curriculum, technologically based or not.

The ESCAPE process for using technology to improve the quality of students' written mathematics

The ESCAPE plan (Explore, Solve, Check, Alternative, Post-Explore) grows out of Polya's original problem solving strategy (Polya, 1945) but has been adapted to incorporate technology into the process. We assume that the students taking Advanced-level mathematics students have already developed some of the elements of Polya's basic scheme: first Understanding the Problem, then Devising a Plan, Carrying out the Plan, and finally Looking Back over their work, because they have been successful in studying mathematics to an advanced level in a highly selective educational system. Such habits, whether explicit or intuitive, must form the backbone of any successful student's work, particularly in a curriculum which is still largely centred on formal manipulation and reproduction of standard techniques. In particular we assume that, given sufficient prompting, students will know how to go about solving the routine (but relatively sophisticated) problems they are presented with. The less able students in the group will find the exploratory aspects of the scheme will help direct their attention towards the solution, and the checking elements should reinforce their work and increase their confidence, or else direct their attention to the existence of anomalies when their work is incorrect. What we wish to illustrate is the additional insights and checking mechanisms that the new technologies make available, to support and enrich the pre-technological approach.

Addressed to a would-be student, the ESCAPE plan can be summarised as follows:

1: EXPLORE the problem before you attempt a solution: working out what it means and getting ideas as to what might happen.

> What sort of answer is expected: a proof, a diagram, a number, an expression?
> Draw a diagram if it seems relevant.

Model the problem: introduce variable names as necessary, and turn the various statements into formulae, equations etc.

If there is a clear direct method according to what you have been taught, you might choose to use it directly and proceed to solve the problem (Stage 2). Otherwise, or if you are unsure, explore further using your calculator in any of the following ways:

- simple calculations of explicit formulas
- guess-and-check or trial-and-error methods
- tabulation of relevant functions
- graphing relevant functions
- use of built-in functions such as an equation solver

Often the problem will concern general algebraic expressions which you can explore by substituting particular values for some of the variables to get an idea of possible outcomes.

When you are ready to start solving the problem, always have an initial "guess" or estimate as to what you might expect the answer to be.

2: Now attempt to **SOLVE** the problem, using the techniques you have been taught, which may or may not involve some use of a calculator.

Ask yourself how confident you are with your "solution". Sometimes you will feel very confident; other times you will be not at all sure that you have got to the right place.

3: **CHECK** the details of the solution when you think you have one:

Using your common sense, your experience, your intuition, is the answer similar to what you would expect?

If the answer is numerical, is the size reasonable? If there are units, do they make sense? Do a mental rough calculation with single digits: does it agree roughly?

Generally, does the answer make sense? Is it realistic? Does it really answer the problem set?

Try to think of a different way of doing the same problem: if you did a calculation, could you have used algebra? Did you work with particular numbers or with a general formula? Did you use a diagram or a graphical method? If you did an exact computation resulting in a string of symbols, check it with an approximate numerical answer.

Doing problems in more than one way makes it easier to check your work confidently AND gives you much more insight into what the problem is about.

4: Try **ALTERNATIVE** methods of solving the problem and compare what happens.

If you used a direct computation, try an algebraic process; if you used an algebraic process or exact computation, try a numerical computation.

You might also consider
- guess-and-check or trial-and-error methods
- tabulation of relevant functions
- graphing relevant functions
- use of built-in functions such as an equation solver

5: POST–EXPLORE the problem after obtaining a solution: re-read and re-think the question, and work out what the solution means in real terms: "interpret" it. Perhaps explore wider problems, such as what happens if some of the original details are changed.

The five-stage **ESCAPE** plan:
- 1: **E**XPLORE
- 2: **S**OLVE
- 3: **C**HECK
- 4: **A**LTERNATIVE
- 5: **P**OST-**E**XPLORE

can be a workable, effective strategy that uses technology to help you ESCAPE from difficulties and achieve success and real understanding in mathematics.

An illustrative example of the use of ESCAPE

Although the five points of the plan are all important, they often overlap and interact in real problems, which can make specific identification of the five stages artificial. We illustrate the philosophy, and the variability of the procedure, in the following two-part example taken from the University of Cambridge GCE A-level paper in Further Mathematics Syllabus C; Paper 1 of June 1991.

The curve C has equation
$$y = \frac{ax^2 + bx + c}{x + d}$$
where a, b, c and d are constants. The lines $x = -2$ and $y = x - 3$ are asymptotes of C, and C has a turning point where $x = 1$. Find the values of a, b, c and d.   [10]
Sketch the graph of C and mark in the coordinates of all turning points and intersections with the coordinate axes.   [4]

Notes on the SOLUTION follow:

EXPLORING

As a preparation, use the calculator to graph curves like C with different values of a, b, c and d in the equation. Look for the features mentioned, especially asymptotes and turning-points. Look for any connexion between the coefficients and the locations of the features.
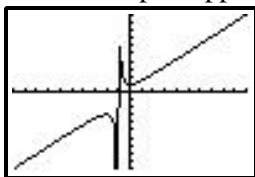
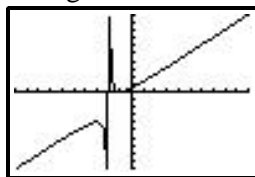Some examples appear in Figs 1 to 3:

Fig 1: Case
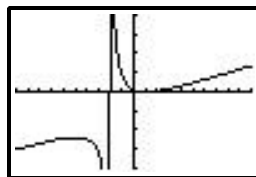a = b = c = d = 1

Fig 2: Case
a = 1, b = 2, c = 1, d = 2

Fig 3: Case
a = 1, b = –2, c = 1, d = 2

Figure 1 was a first attempt; Figure 2 successfully located the vertical asymptote at x = –2, and Fig 3 successfully located the turning point at x = 1. Clearly there are more possibilities to explore!

[Figures 1 and 2 are drawn in the Standard Window of the calculator used, with –10 = x = 10 and –10 = y = 10; the "spike" is typical of the way vertical asymptotes appear in a limited screen display. In the case of Figure 3 the left hand portion did not appear in the Standard Window and the screen was adjusted to show –20 = y = 20. The "exploration" in this instance was fairly precisely directed; many students would need to examine many additional combinations before achieving the level of "success" illustrated in Figure 3, or might find it without a lot of insight as to how it was done.]

SOLVING:

TYPICAL PENCIL-AND-PAPER WORKING (FIRST PART)

(1)   If the denominator of the equation is x + d, there will usually be a vertical asymptote at
        x = –d. Since C has a vertical asymptote x = –2, the value of d must be d = 2.

(2)   If we divide the denominator into the numerator, we will get

$$y = \frac{ax^2 + bx + c}{x + d} = mx + n + \frac{p}{x + d}$$

        for some constants m, n and p; and then the sloping asymptote will be y = mx + n. Since
        the sloping asymptote is y = x – 3, the equation of C must be

$$y = x - 3 + \frac{p}{x + 2}$$

        for some constant p. It still remains to find p.

(3)   We can now differentiate:

$$y' = 1 - \frac{p}{(x + 2)^2}$$

        and use the fact that C has a turning point at x = 1 to deduce that y'(1) = 0, and thus

$$0 = 1 - \frac{p}{9}$$

so that $p = 9$.

(4)   Now we recombine to get the equation of C in the original form:

$$y = x - 3 + \frac{9}{x + 2} \;=\; \frac{(x - 3)(x + 2) + 9}{x + 2} \;=\; \frac{x^2 - x + 3}{x + 2}$$

and deduce that $a = 1$, $b = -1$, $c = 3$ and $d = 2$.

CHECKING (FIRST PART)

Now graph this function on the graphics calculator and check that it has all the features expected. Standard windows should show at least a portion of the graph; there will be a turning point close enough to $x = 1$, and the asymptotes expected should look reasonable. Depending on the window chosen, there will be either a "false" vertical asymptote or a clear gap at $x = -2$ (Fig 4).

Also draw the sloping asymptote as $Y = X - 3$. This should seem to be a good fit to the curved graph. Extend the window to see a range of the order of $-10 = x = 10$, $-20 = y = 20$ or $-30 = y = 30$. The essential features of the graph should now be apparent (Fig 5, $-30 = y = 30$). Use the calculator's minimum function to check that the turning point is at $x = 1$ as required (Fig 6).
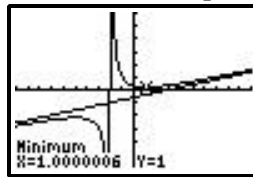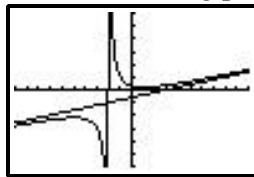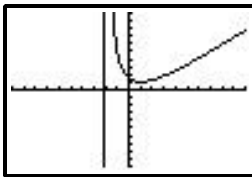

Fig 4: Standard window   Fig 5: Adjusted window   Fig 6: Minimum located

ALTERNATIVE METHODS: we have covered this in effect by using both algebraic reasoning and graphical computing to produce compatible pictures.

TYPICAL PENCIL-AND-PAPER WORKING (SECOND PART)

The calculator gives us a reasonably clear graph, but you will be expected to "show working" to justify each feature required. In this part we switch constantly from manual **SOLVING** to calculator **CHECKING**; indeed it is natural for students to first see the required features on their calculator and then justify them by algebraic reasoning.

(1)   **Intersection with y-axis:**

CALCULATOR: use the tracing cursor or a built-in value function with $x = 0$ to get $y = 1.5$ (Fig 7).

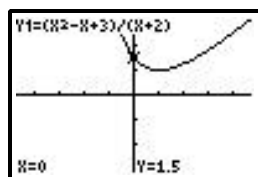PENCIL-AND-PAPER: in the equation put $x = 0$ to get $y = \frac{3}{2}$; $(0, \frac{3}{2}\,)$ is the y-axis intercept.

Fig 7: y-axis intercept

(2) Intersections with x-axis:

CALCULATOR: from Figs 7 & 8 it is fairly clear there are no intersections (given a basic knowledge of rational functions we can assume the graph does not bend further outside the window shown in Fig 8).
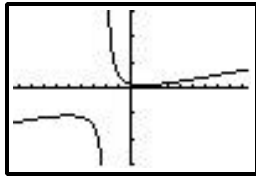

Fig 8: No x-axis intercepts

PENCIL-AND-PAPER: in the equation put y = 0 to get $x^2 - x + 3 = 0$.

Using the quadratic equation formula gives $x = \dfrac{1 \pm \sqrt{-11}}{2}$ which indicates no real roots and therefore no

intersections. The complex roots $\dfrac{1}{2} \pm i\dfrac{\sqrt{11}}{2}$ have no significance on the graph.

(3) Vertical asymptote:

CALCULATOR: depending on the window chosen, vertical asymptotes appear either as spaces in the graph (when there is a pixel centred on the x-value where the function is undefined, Fig 9) or as a sharp near-vertical line or spike (when the function is defined at each pixel centre, Fig 10). One of these features should clearly occur at and near x = −2 in this case.
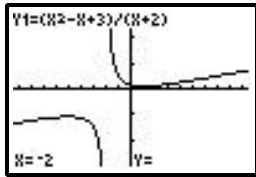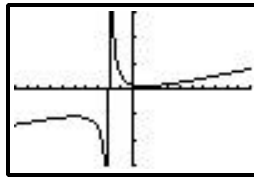

Fig 9: Missing value   Fig 10: False asymptote

PENCIL-AND-PAPER: the vertical asymptote(s) occur when the denominator of the function is zero: here when x + 2 = 0, or x = −2.

(4) Sloping asymptote:

CALCULATOR: without algebra it is difficult to get the asymptote correctly, although "educated" guessing might find it. In this case the coefficients (a = 1) imply that the equation will be y = x + k for some k, and successive guessing will probably lead to y = x − 3 as a plausible but not proven equation (Fig 11). If desired, zooming out will confirm the asymptotic behaviour more vividly.
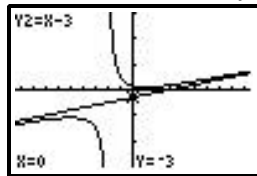

Fig 11: Sloping asymptote?

PENCIL-AND-PAPER: really has to be done algebraically as in the first part.

[Note that none of the illustrations has equal scale units on the two axes, so that slopes of lines and shapes of certain curves appear "wrong". Students experienced in the use of graphical calculators should be used to this problem: if desired, the scales can be "squared", but usually with loss of detail in other areas.]

(5)   Turning points:

CALCULATOR: There is clearly a minimum point in the region of $x = 1$. Use of the calculator's minimum function places it (correct to about 5 decimal places) at $x = 1$, $y = 1$ (Fig 12). Similarly the calculator's maximum function locates the maximum point at $x = -5$, $y = -11$ (Fig 13).
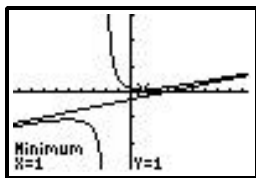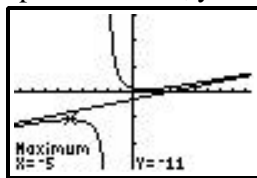

Fig 12: Minimum point   Fig 13: Maximum point

PENCIL-AND-PAPER: Take $y = \dfrac{x^2 - x + 3}{x + 2}$ or $y = x - 3 + \dfrac{9}{x - 2}$ and differentiate:

$$\frac{dy}{dx} = \frac{(x + 2)\,(2x - 1) - 1\,(x^2 - x + 3)}{(x + 2)^2} \qquad \text{or} \qquad \frac{dy}{dx} = 1 - \frac{9}{(x - 2)^2}$$

Set this to zero, which in the first form means the numerator is zero:

$$\frac{dy}{dx} = 0 \ \text{ when } \ (x + 2)\,(2x - 1) - (x^2 - x + 3) = 0 \qquad \text{or} \qquad 1 = \frac{9}{(x - 2)^2}$$

$$(2x^2 + 3x - 2) - (x^2 - x + 3) = 0 \qquad \text{or} \qquad (x - 2)^2 = 9$$

$$x^2 + 4x - 5 = 0 \qquad \text{or} \qquad x - 2 = \pm\,3$$

$$(x + 5)\,(x - 1) = 0$$

$$x = -5 \text{ or } x = 1$$

For each x-value we then calculate the matching y-value to locate the two turning-points at $(1, 1)$ and $(-5, -11)$. To confirm their nature we find the second derivative:

$$\frac{dy}{dx} = \frac{x^2 + 4x - 5}{(x + 2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{(x + 2)^2\,(2x + 4) - 2\,(x + 2)\,(x^2 + 4x - 5)}{(x + 2)^4}$$

$$= \frac{(2x + 4)\,[(x^2 + 4x + 4) - (x^2 + 4x - 5)]}{(x + 2)^4}$$

$$= \frac{18}{(x + 2)^3} \qquad \text{[or directly from } \frac{dy}{dx} = 1 - \frac{9}{(x - 2)^2} \ \text{]}$$

At $x = 1$ the second derivative is positive, so the point is a minimum; and at $x = -5$ it is negative, so the point is a maximum. Incidentally we note that when $x < -2$ the second derivative is always negative, so the

curve is concave down; and when x > 2 the second derivative is always positive, so the curve is concave up.

All of this information can now be used to draw a good sketch graph of the function, which will be like that seen on the calculator (Fig 14). **CHECKING** has already taken place in that we have worked out each feature both manually and on the calculator. **ALTERNATIVE** methods have likewise already been illustrated.
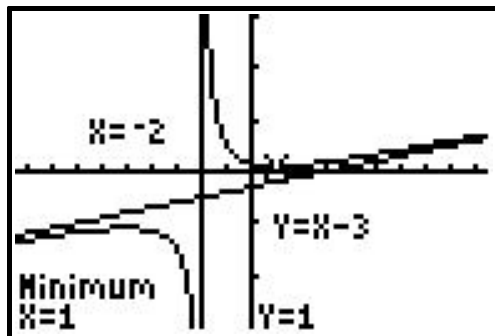

Fig 14: Overall picture.

POST-EXPLORING

The original form of the equation invites us to consider the general shape of the graph of a rational function which is a quadratic function divided by a linear function

$$y = \frac{ax^2 + bx + c}{x + d}$$

Students can use their calculator to explore these as fully as they wish. There are four features that can alter the shape:

(1)   The numerator may have two distinct roots, or one repeated root, or no real roots: this will affect the x-axis intercepts in particular.

(2)   Where there are two roots, it is significant whether the point $x = -d$ comes between the two roots, or to the left of both, or to the right of both. It is assumed that neither root is at $x = -d$ (if it is, the graph degenerates into a straight line with a single missing point).

(3)   Apart from this, the value of d affects the location of the vertical asymptote and hence the y-axis intercept, if any.

(4)   When $a > 0$ the sloping asymptote slopes up to the right, and the reverse when $a < 0$.

Altogether (apart from the degenerate case) there are four different basic shapes and two dozen or so different combinations of shapes and intercepts! A selection of these are shown in Figs 15–19.
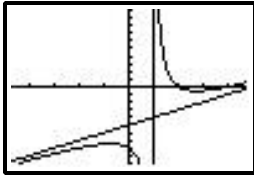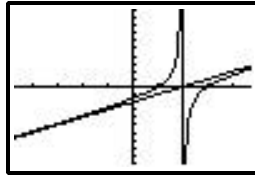
Fig 15: Case
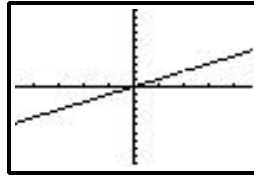a = 1, b = –6, c = 8, d = –1

Fig 16: Case
a = 1, b = –4, c = 3, d = –2

Fig 17: Degenerate case
a = 1, b = –2 c = 0, d = –2

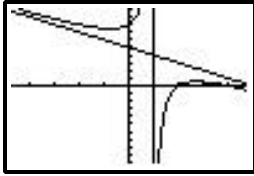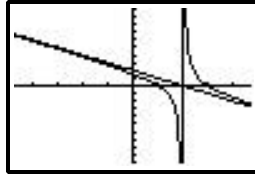Fig 18: Case
a = –1, b = 6, c = –8, d = –1

Fig 19: Case
a = –1, b = 4, c = –3, d = –2

These examples illustrate the potential of good technology to develop understanding beyond that available with pencil-and-paper techniques alone. With careful graphing (coupled with algebraic analysis) one can "see" the dynamic effect of the component parts of the function on the shape of the graph, and even incorporate the degenerate case as an intermediate stage.

Discussion and conclusion

In this paper we have shown how a technology such as the graphics calculator can be integrated into a student's problem solving strategies to enhance the quality of the student's written mathematical work, even when it contributes little to the direct solution of the problem. One might ask, why bother, if the able student is capable of generating the required solution without such assistance? Our response would be that, used in this way, the graphics calculator is able to give the student particular insights into the problem solution, most often algebraic in nature, through alternative graphical and numerical representations that would not be available to the student otherwise. In addition, the ease with which the calculator is able to generate particular cases of the general solution provides a ready mechanism for the student to check the validity of the answers they obtain, in a way that is not feasible with standard pencil and paper techniques. This has two advantages: it encourages good mathematical practice, and, if the goal is maximizing an examination score, the graphics calculator enhances the ability of the student to detect the sort of minor errors that even the best of students make under examination conditions.

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