What is “doing mathematics” now that technology is here?

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Abstract

All over the world people say that mathematics is difficult and a problem. "Doing mathematics" is widely perceived by the general population as carrying out mechanical algorithms. However, it is much more than this. Rapid advances in technology give the opportunity to broaden perceptions of what doing mathematics is now, to include its role in communicating ideas and solving real problems. Symbolic algorithms and language are only one aspect of doing this, but unfortunately this aspect is restricted to a minority of people. Many find this symbolism too demanding and too abstract, and consequently cannot use mathematical ideas which would be helpful to them.

Now that there is a wide range of powerful mathematical and other technology, both cheap and expensive, which support the mechanics of "doing mathematics" (such as CAS), we might ask what skills and knowledge remain or become essential.

Problems are often stated in words or pictures, so it certainly remains essential to be able to communicate in the sense of being able to translate from words into mathematical language and vice versa - in other words to formulate problems, and to interpret their solution. In fact while mathematicians require formal proof, most people will be happy with a persuasive argument. Thus the mathematical communication must be appropriate to the audience. Technology will always give an answer but just as important is to make sense of that answer and to validate it.

The whole range of technology is having an impact on mathematics, and not only specific mathematical machines and packages. Opportunities to apply mathematics are enhanced by digitised film, recorded sound, and easily used data gathering devices, now being integrated through packages such as Motion, Sound, TI-Interactive and the CBL/CBR. Thus numbers and pictures generate and motivate engagement with mathematical problems, as well as suggesting ways in which answers can be presented appropriately.

In this paper we describe some of our experience in modifying our practice to accommodate technology. We also raise some of the questions we are still having to ask, such as what constitutes the "basic skills" of mathematics, including basic symbolic manipulation skills.

Introduction

Most people in the world are not mathematically competent. Full mathematical competence seems to be the prerogative of a small rapidly declining cohort of academics. A few, some might say misguided individuals would like to keep up to date with developments in mathematics but they are faced with daunting problems. The highly abstract notation and communication language used by “doers” of mathematics make any expedition into mathematics difficult. Mathematicians can be their own worst enemy in this respect and many almost make a virtue of perpetuating this complexity, arguing that the subject and its language are inherently difficult.

Opposed to this, it is apparent that nevertheless, mathematics and mathematical thinking are valuable in many contexts, but for the ideas to be exploited to the full they need to be accessible. At a time when, in the UK at least, some universities are closing mathematics departments, we in the academic mathematics community have a choice. Do we want to be an inward looking and endangered species, or do we make every effort to make mathematics accessible, and to send out the message about the value of mathematical thinking?

Attitudes to this issue differ widely. Some emphasise the narrow rote algorithmic aspect of mathematics by testing students with mathematical tests from the past, purporting to show how mathematical standards are not as high now as they were in past ages (usually when the tester was tested!). These students of mathematics are faced with rote memorising of multiplication tables or algebraic technique and often have the feeling that mathematics is little more than an invariant body of knowledge that must be passed on from generation to generation. Scientists and engineers broadly think of mathematics as a warehouse of ready-to-select formulae, theorems and techniques to advance their own theories, and all that is needed to use this warehouse is to replicate an algorithmic process. These narrow views are indeed what most people regard as mathematics.

Missing, however, is how mathematics is constantly being used, discovered, created, discussed and critically affected by changes both in society, and especially in technology. The world is constantly changing and the relevance of skills from the past and in particular skills in mathematics is also changing, and we must constantly remind ourselves of this. We have all forgotten how to hunt and kill mammoths because there is no need to have this skill. So it is with mathematics - the initial skill base of students arriving at university is different from the past, and at a time of rapid change in society and technology, the question of what constitutes "doing" mathematics requires regular review. This is the subject of this paper.

"Doing" mathematics can be many things to many people. In these days of the perceived need for increasing (and publicly proven!) excellence of provision in mathematical education, the assessment of rote processes that are easy and cost effective to test provides a simplified, indeed over-simplified issue for rhetorical use by politicians. The problem here is that testing rote practices such as tables or basic algebraic technique is only one very small part of what doing mathematics is. To judge by this alone is like judging a footballer purely by their ability to head a football. All the rich experiences in the game and its culture cannot be obtained by concentrating solely on just one skill.

Going for a *SONG*

The authors' own view is that "doing" mathematics is something much wider than rote practice of algorithmic skills, and it is important to get this message across to a wider population. We encourage our students to "go for a ***SONG***" [1], [2]. This makes the point that students should address the richness of a mathematical concept, by combining ***S***ymbolic, ***O***ral, ***N***umeric and ***G***raphic approaches, and by making use of whatever technology comes to hand. We are applied mathematicians and modellers by inclination, so we also aim to make mathematical concepts emerge from experience of the world, but this does not necessarily restrict the scope of our thinking.

"*O*" for *Oral*

We expand first on the ***O*** because the fundamental activity of "doing" mathematics is that of communication. Even abstract mathematical ideas grow from our knowledge of the world, and experience suggests many problems. These may take the form of a situation described in words:

> *"When ploughs were drawn by horses they generally had only one ploughshare, or blade. Nowadays tractors pull ploughs with four, five or even more ploughshares (for use in one direction: most ploughs*

have two sets of ploughshares, so that the tractor can reverse at the end of every row of ploughing). When a large field is to be ploughed, having more ploughshares on the plough saves time; on the other hand, the plough costs more when it has more blades, and in addition it presumably requires more fuel to pull more ploughshares, especially on heavy soil. Each farmer has to decide what sort of plough to buy."

Alternatively there may be a video recording of an occurrence (Figure 1), or a noise (Figure 2):

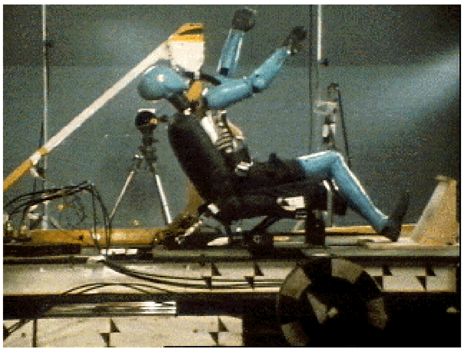

Figure 1 (produced in Multimedia *Motion*, © Cambridge Science Media 1996 [3])

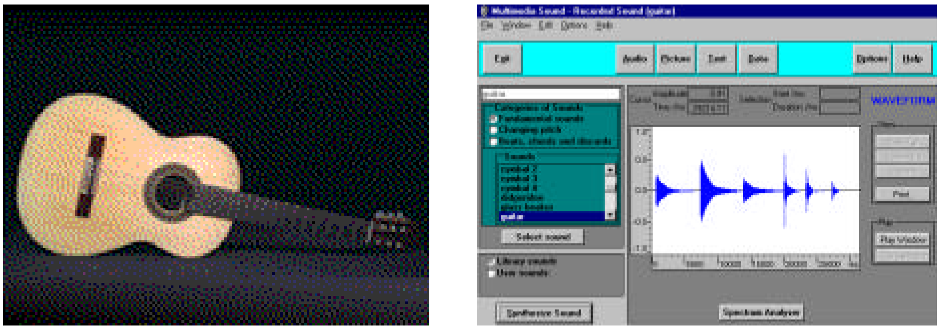

Figure 2 (Produced in Multimedia *Sound* 1.0 © Cambridge Science Media 1996 [3])

Communication is a key aspect of developing a mathematical formulation and understanding of such situations, and this is part of doing mathematics. Identifying the parameters, probably by discussion with a variety of people, is only an early stage of expressing all these various situations in some formal language, to model, simulate and try to understand the factors involved. The formal language is essential if we are to call on and use the powerful benefits of modern technology. It is worth mentioning that the form of this mathematical language is affected by having to communicate with machines. Thus for instance we already notice that our students, heavily influenced by their graphic calculators, commonly use an ordered pair notation for complex numbers alongside the standard $a + ib$ notation. It is also worth saying that notation is not a trivial issue, as evidenced for example by the early days of calculus, and the competing notations of Leibnitz and Newton.

The formal language of mathematics is not widely understood outside the mathematics community, but our solutions and conclusions must usually be communicated to a wider audience (bosses, non-mathematical colleagues, funders, etc.) and again this is still part of doing mathematics. The main point is that no matter what the analysis of the related problems gives, it has to be communicated in the appropriate way to the appropriate audience, and that audience will usually be less well qualified mathematically than the presenter. This, we feel, is a critical problem in the area of mathematics. Some of our students do not perceive the importance of communication immediately: "I did not come here to write essays, I want to do mathematics". Similarly many academic mathematicians spend most of their working life surrounded by peers with a similar amount of knowledge and the same taxonomy.

The situation concerning wide communication of mathematical ideas is further complicated not only by the appropriation of simple, everyday words to be used in unexpected ways and assigned to specific, technical meanings to express abstract concepts. There is also the tendency, present in all such professions as ours, to mystify by the use of a highly restricted code; e.g.

> *"**S**ymbolic Itô Calculus is the name for implementation of Itô calculus (stochastic calculus for continuous semimartingales) in various computer algebra packages"*

(Sentence picked at random, and with respect, from one of many course web sites. Other examples abound!)

To continue the football analogy mentioned earlier, communication is a necessary part of the game of football - we must possess good and appropriate skills, but we must also communicate with our team in the appropriate language. Good goalkeepers have loud voices and meaningful gestures.

In the search for clear understanding and unambiguous communication concerning mathematical treatments, we are led to descriptions which may involve translation into and out of some sort of formal language (***O***ral). However we are also led to ***N***umeric (including perhaps digitisation) or ***G***raphic (pictorial) descriptions.

"*N*" for *Numeric*

The world is comfortable with the notion and notation of ***N***umber. Multiplication tables are practiced and processed, money is calculated, football crowds are counted, and the number of research papers counted! However it is worth noting that a prime (one of many words appropriated by mathematicians and given their own meaning) part of mathematics now is discrete mathematics, which had little significance in mathematics before technology exploded onto the scene. Reality is now collected discretely (another of those redefined words) as computing machines cannot handle continuous processes. Even the image of the Mona Lisa is reducible to a collection of numbers defining a bitmap image now (see Figure 3) …

M =

	1	2	3	4	5	6	7
1	143	147	152	151	152	152	151
2	154	138	154	143	147	152	159
3	147	136	145	152	152	154	151
4	151	145	158	154	142	151	151
5	149	149	163	161	151	161	165

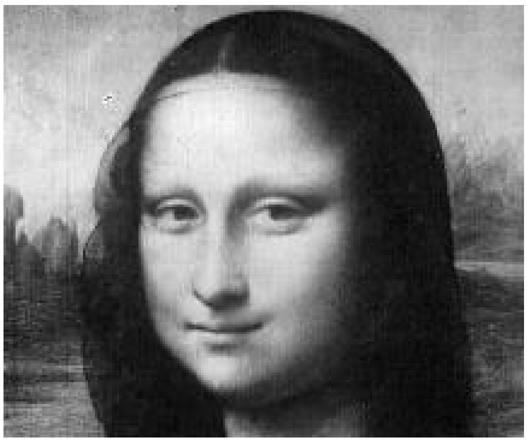

Figure 3

… and the behaviour of the crash test dummy earlier is expressible in terms of gathered discrete data from the "reality" of the video (see Table 1).

The notion of "doing" mathematics has to accommodate this critical change in the role and interpretation of ***N***umbers.

Table 1 (produced in Multimedia *Motion*, © Cambridge Science Media 1996 [3])

Seat belt test:				
	Data Car		Data Dummy	
t/s	x/m	y/m	x/m	y/m
0	0.113	0.316	0.09	1.321
0.011	0.266	0.319	0.246	1.321
0.022	0.419	0.316	0.399	1.321
0.032	0.559	0.326	0.556	1.321
0.043	0.675	0.326	0.709	1.324
0.054	0.775	0.329	0.858	1.328
0.065	0.858	0.329	1.011	1.328
0.076	0.918	0.333	1.161	1.331
0.086	0.972	0.333	1.298	1.311

The connection between numbers and pictures, made particularly powerful now by the use of technology, is an essential part of mathematics, which enhances communication of answers and results to all. Furthermore, developments such as the rapid spread of the mobile phone, and processed and compressed images, emphasise again the role of discrete ***N***umerical processes in the "doing" of mathematics as used outside the immediate world of mathematics itself.

However numbers are not everything!

> "“Can you do addition?” the White Queen asked.
>
> “What’s one and one and one and one and one and one and one and one and one and one?”
>
> “I don’t know,” said Alice. “I lost count.”" [4]

“G"* for *Graphic

The representation of mathematical ideas to any audience is enriched and enhanced by visualisation. This is the key to unlock understanding, since graphs and pictures are more easily interpreted than symbols by people at all levels of mathematical sophistication. Obvious examples are the pictogram and pie chart shown in Figure 4, produced on a TI-73 graphic calculator.

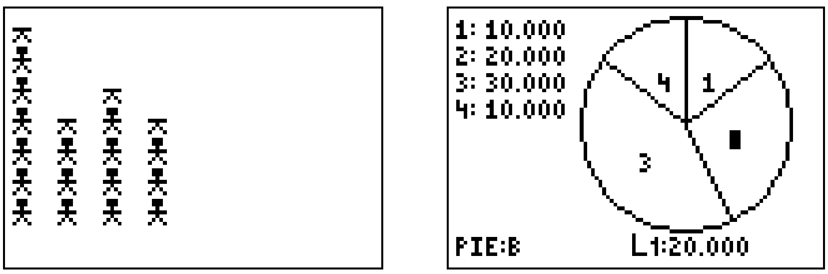

Figure 4

Interpretation of mathematically more complex processes, such as a bungee jump, can also benefit from suitably rich and powerful visualisation. In an earlier article [5] the authors collected real data from a bungee jump using a CBL™/CBR™ through a graphic calculator. Analysing the situation [6] gave a 3D plot of the displacement of the poor soul (actually a drink can) tied to the bungee rope, against both time and variable spring stiffness, as shown in Figure 5.

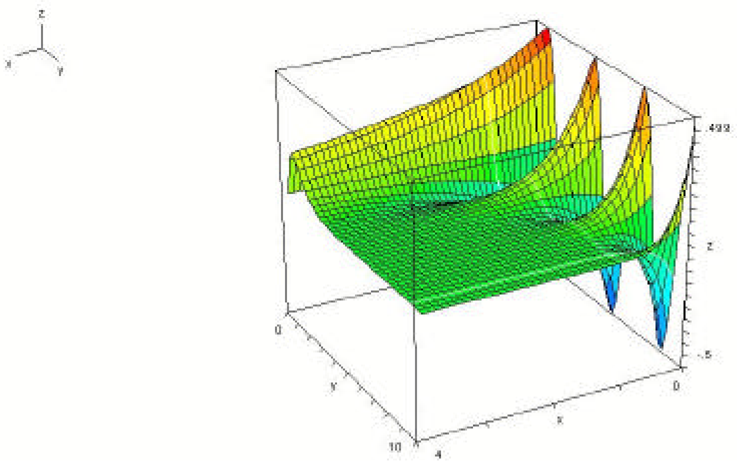

Figure 5

Figure 5 was produced (and can be spun) in Derive version 5. Few can analyse and process this situation symbolically, still less explain the symbolic solutions, but many can benefit from trying to make sense of the rich, graphical representation of the system behaviour.

The exercise here then becomes an interpretation of this ***G***raph in terms of the behaviour of the system. The fact that the picture can also be rotated simply adds to the impact, as well as helping with investigation of the system.

The " *S* "

To the mathematically uninformed (and many mathematically informed!) the ***S***ymbolic aspect is their major perception of mathematics. We deal with it last here for that reason. It is easy for *mathematicians* to use precise, compact, symbolic notation. This comfortable, specialised language enables "simple" solutions of problems. It seems it is easier and more efficient to state or write down an equation or some shorthand mathematical expression than to express the idea in words. However it is commonplace also to assume or hope that any student (or broader) audience has equal facility with this language, and perhaps to complain when they do not. A series of reports in the UK [7], [8] has indeed concentrated on and identified what are perceived as increasing shortcomings in this respect in UK students newly arriving at university.

Obviously highly relevant in this discussion about the place of symbolic manipulation in mathematics, is the continuing and rapid development of Computer Algebra Systems (CAS). These have been around for many years and are now inexpensive, portable and commonplace to varying degrees in academic life. For instance the TI-89 [9] has a CAS available, and also has a *Script* language as a "language to communicate with machines". The example screens in Figure 6 show some of the facilities available in this machine which is superficially, and to the uninformed eye, indistinguishable from an ordinary graphic calculator. One can only assume that examination invigilators will not be uninformed.

Given that such technological power and assistance exists and is commonly affordable, this must raise the question of how it will affect the general view of what constitutes "doing" mathematics. As common perception now places the emphasis on mechanical enactment of symbolic and numerical algorithms, which can now be done by machine, perhaps that emphasis can move to how we can communicate ideas and requirements properly, not only with people, but also with machines. The scripting language illustrated in Figure 6 raises interesting possibilities here.

However there is a caveat. We are aware that our enthusiastic perspective may be affected by the fact that we were as young people extensively and intensively subjected to "drill and kill", in order to develop the traditional algebraic algorithmic skills. That must affect our view of the possibilities raised by CAS. How do young people without our experience view this? How many of the old skills do they need to be competent with, and to what degree, before they can use CAS comfortably and competently? We feel the need for more research.

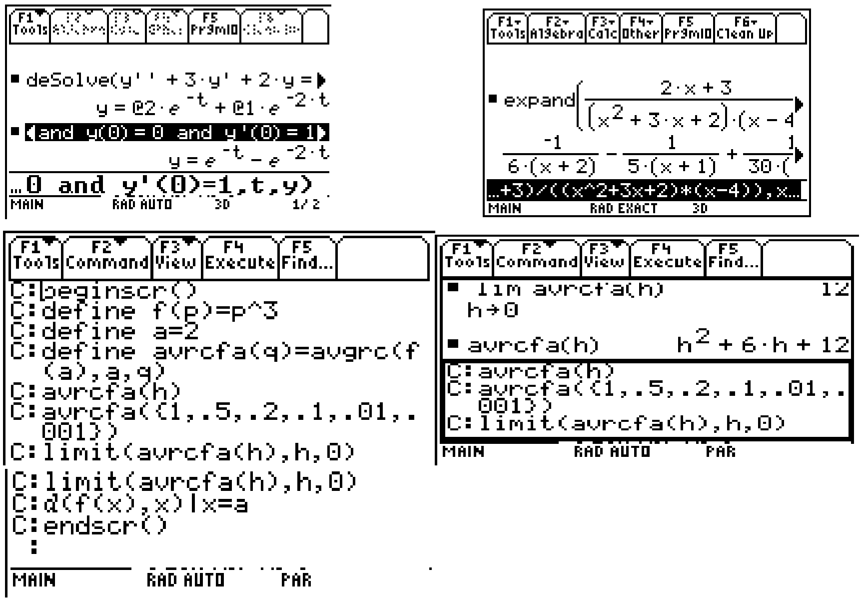

Figure 6

In discussing the role of CAS then, we find ourselves returning to the ***O*** for ***O***ral: it is necessary to communicate - in this case with machines. It is interesting to speculate about the future, particularly given the literal meaning of the word "***O***ral". Verbal communication is of course paramount (at the time of writing this paper over 50% of people in the UK have mobile phones - and that includes children!). One can of course already speak to a computer, but in trying to verbalise mathematics, significant obstacles remain. One only has to try the simple experiment of asking the general population (or indeed new first year students) to give the answer to the question: “What is one plus two times three?” The very existence of *two* well-supported responses betrays the lack of awareness of the importance of linguistic structure and rules in mathematical language. When shown the error of their ways the reply is often a defiant “I could never do maths at school” - and then begins a catalogue of blame!

Technology which deals with *SONG*

Technology is now appearing which reflects the need for a broad and integrated approach to mathematics. For example recently released in the US is the TI-Interactive™ software package [10]. This integrates the range of approaches to mathematics into a learning package that combines standard PC facilities (CAS (***S***), word processor (***O***), spreadsheet (***N***)), linked to hand held technology and to the internet. Students’ learning may be enhanced through interactive lessons that encourage exploration, visualisation (***G***), data analysis (***N***) and writing (***O***). Such packages can encourage communal working on mathematics problems, and indeed provide a step towards forming a community of users. They can also encourage a broad and self-checking approach to problem solving. In the output shown in Figure 7 for example, a problem is defined in words, an algebraic formulation provides the answer, the spreadsheet reinforces this answer, and this is then checked with a hand held device (a TI-83 screen read into the package).

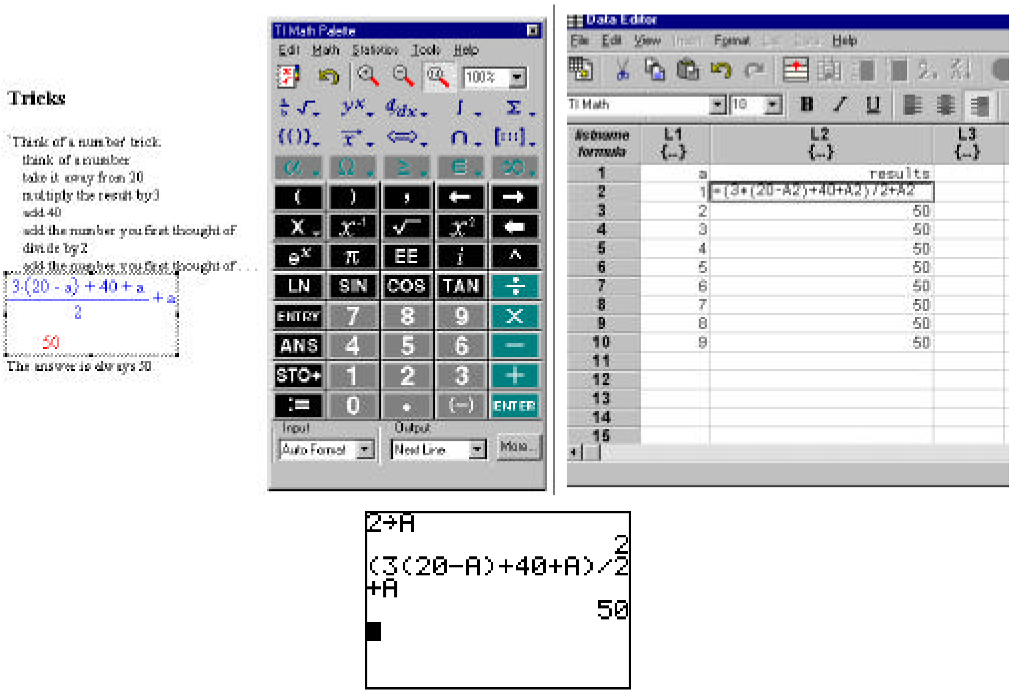

Figure 7

The user can make the connection with reality through the package in a variety of ways. Figure 8 shows the wealth of data available for modelling from the associated web site. Alternatively one can collect personal data using a data logger such as CBL^{TM} with the appropriate probe or a CBR^{TM}, or gather data from less accessible places using a package such as Motion [3].

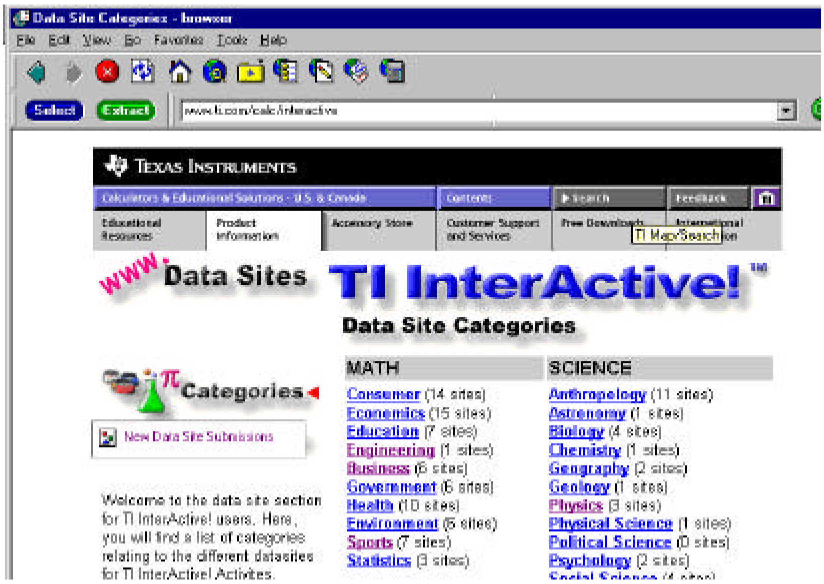

Figure 8

In such an environment it is very difficult to get a wrong answer in the narrow sense, and the emphasis can move towards understanding and formulating the problem, and making sense of and communicating the solution.

Conclusion

In this paper, we have discussed perceptions of the process of "doing" mathematics now that technology is here. We have raised many different areas for discussion. We believe that both the mathematics curriculum and the associated assessment practice must be overhauled to take account of the rapid advances in technology, and we have designed the mathematics degree at our own institution with this in mind.

Furthermore, in earlier papers [1], [2] we have also noted the globally recognised issue of key or transferable skills, which may be categorised as communication, solving problems, working in groups, using IT, improving one's own learning, and application of number. Mathematics in its broadest form as discussed here allows many opportunities for students to develop all of these skills.

The heart of mathematics remains central to human thought as it should, with its rigour, its logical thought processes, and what may be seen as its austere beauty by those few lucky enough to appreciate it. However, mathematical ideas are too important to be restricted to the few. It is incumbent upon those of us in the world of mathematics to make every effort to demystify the idea of "doing" mathematics, and to make the ideas as widely accessible as possible. Technology provides an unprecedented and exciting opportunity to carry this forward.

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