

# Computing the logarithm and the value of the argument of a Gauss Integer

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## Abstract

I have written several practical educational programs based on computer assisted instruction to help high school students understand abstract mathematical concepts <sup>(1,2,3,4)</sup>. In this case, I deal with the logarithm and the argument of a Gauss Integer.

It is generally difficult for students to obtain these values because they do not learn calculus correctly. They usually obtain them using a number table without gaining a clear understanding of the underlying principles. This paper explains how to calculate these values without using calculus and a number table, but with a simple computer assisted instruction program.

## 1. The idea of involution

We often use the idea of involution to analyze the relationship or nature of numbers. In this study, the idea of involution refers to this analysis and is used to compute the logarithm and the value of the argument of a Gauss Integer. Although they seem to be completely different matters, they share the same structure, making it possible to obtain them using the idea of involution. The details will be explained in the following chapters.

Expansion can be used to understand the idea of involution. For example, to calculate  $1.2 \times 1.7$ , we multiply (i.e., expand) both numbers by 10. After calculating the value of  $12 \times 17$ , we divide it by 100 to obtain the answer. High school students need an idea similar to that of expansion. For example, in order to select which one of  $\sqrt[3]{4}$  and  $\sqrt[4]{6}$  is larger, we raise these numbers to the 12th power. (i.e.  $(\sqrt[3]{4})^{12} = 256$ ,  $(\sqrt[4]{6})^{12} = 216$ ) Thus, proving that  $\sqrt[3]{4}$  is bigger than  $\sqrt[4]{6}$ . This is the example of the idea of involution.

## 2. Computing the logarithm value using the idea of involution

There are several ways to obtain the approximate value of  $\log_{10} 3$ . If we can proof the inequality  $10^{47} < 3^{100} < 10^{48}$ , we can prove that  $\log_{10} 3 = 0.47$ . Moreover, if we can find an integer  $m$  such that for an integer  $n$   $10^m < 3^n < 10^{m+1}$  holds, then we can prove that  $\frac{m}{n} < \log_{10} 3 < \frac{m+1}{n}$ . If  $n$  is large enough,  $\log_{10} 3$  is close to  $\frac{m}{n}$  because  $\lim_{n \rightarrow \infty} (\frac{m+1}{n} - \frac{m}{n}) = 0$ .

Thus, we can grasp the relationship between 3 and 10 (or any other pair of numbers) by comparing their involutions. However, it is impossible to obtain the value of  $3^n$  for large value of  $n$  using a pocket calculator. On the other hand, it is very easy to check this inequality using a computer.

### 3. PC-Log : a program for computing the logarithm value

I wrote the Visual Basic program named “PC-Log” to obtain the value of a logarithm. The main window is shown in figure 1. Involutions of the bases 10, 2, 3 and 6 are shown on the screen in increasing order. If we click one of these involutions, we can see its value in the lower right box. In addition, we can see its exponent in Japanese and English. In Japanese, there are numerals of exponent up to  $10^{75}$  (Sen Taisu) and in English, up to  $10^{122}$  (Hundred Novemtrigintillion). To see the big numbers, the scroll bar is moved. In figure 2, we can see  $2^{100}$  (called nonillion in English). We can also see that  $2^{100}$  is between  $10^{30}$  and  $10^{31}$ . So the value of  $\log_{10} 2$  is close to 0.30. Similarly, we can compute the value of  $\log_{10} 3$  and  $\log_{10} 6$  by comparing them with the involutions of 10. Figure 3 shows the case for  $\log_{10} 6$ . We can see that  $6^{100}$  is between  $10^{77}$  and  $10^{78}$ . Of course, it is not necessary to compute the 100th power of these bases.

We can change the source base, 1st base, 2nd base and 3rd base respectively by clicking the list boxes in order to compute other logarithm values.

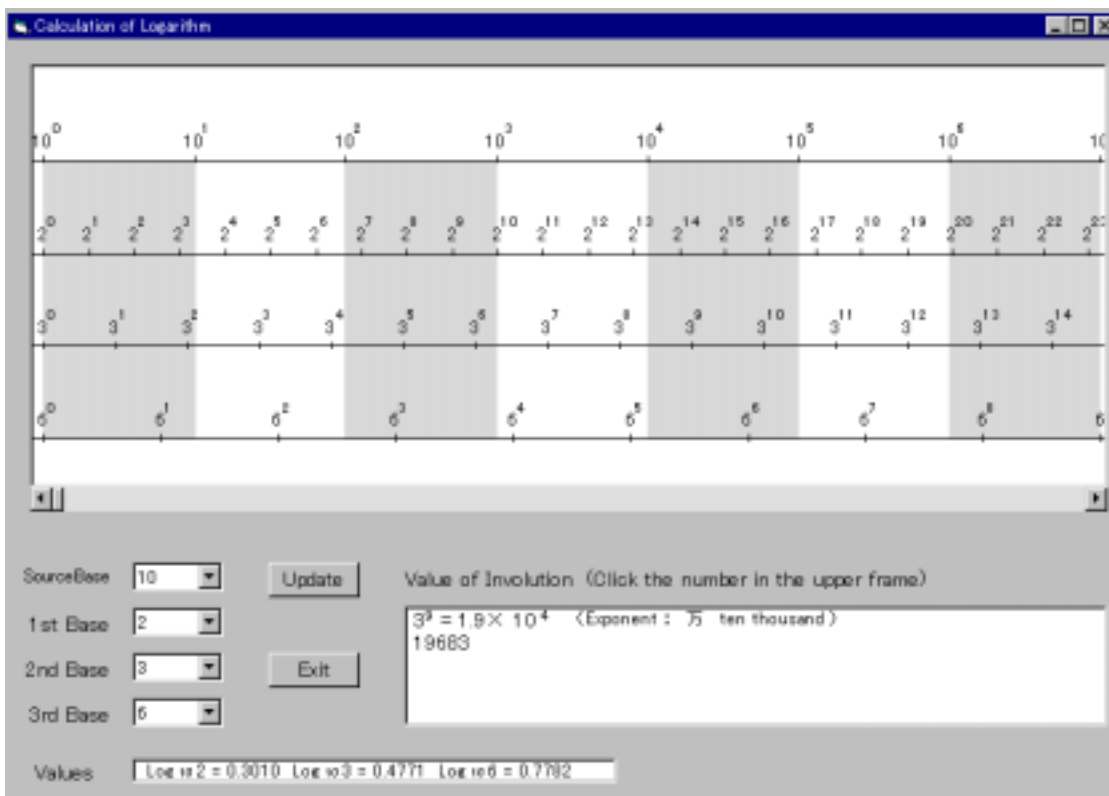


Fig 1 Main window of the PC-Log program which computes the logarithm value

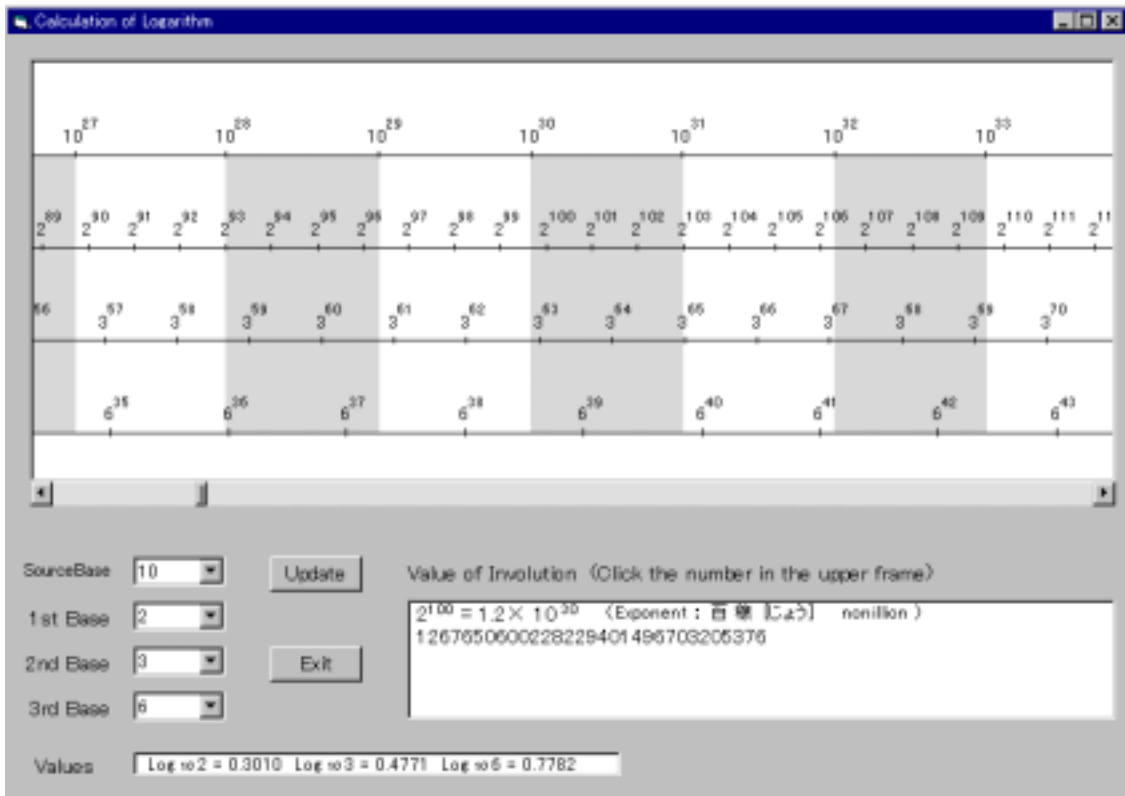


Fig 2 Example showing that  $2^{100}$  is between  $10^{30}$  and  $10^{31}$

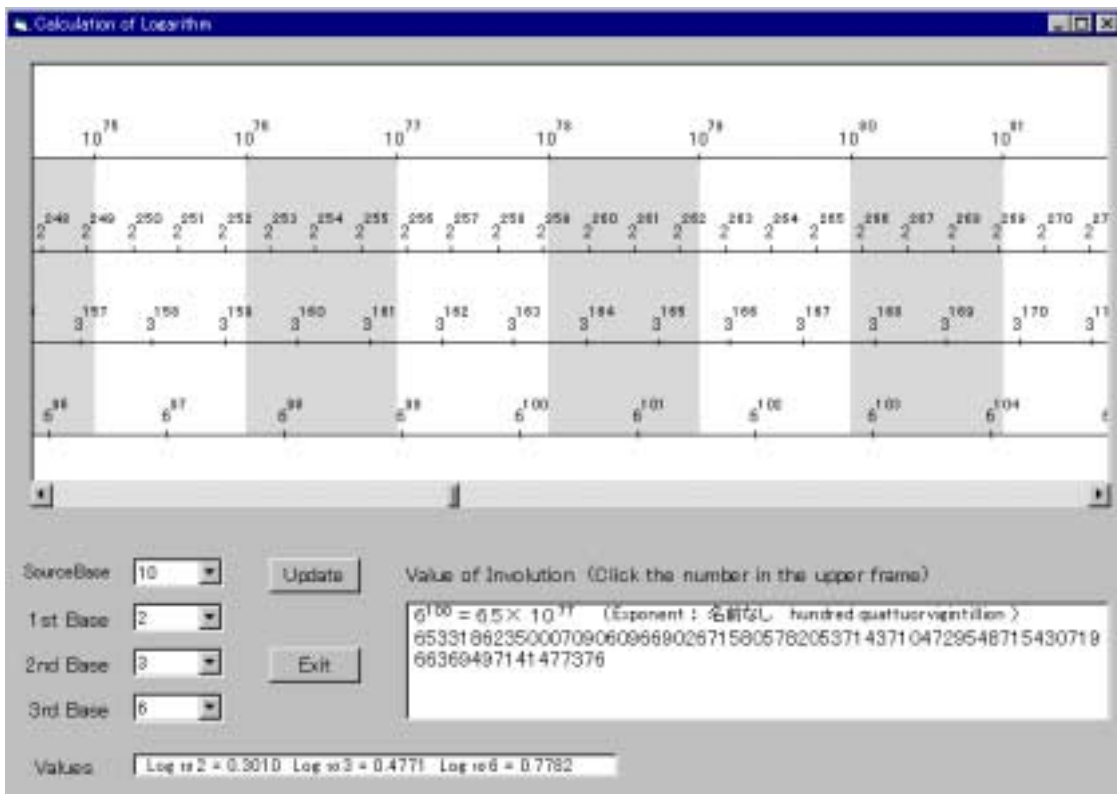


Fig 3 Example showing that  $6^{100}$  is between  $10^{77}$  and  $10^{78}$

#### 4. Computing the value of the argument using the idea of involution

We can also obtain the approximate value of  $\arg(a+bi)$ ,  $a$  and  $b$  integers using the idea of involution. In the case of  $z=2+3i$ , we can find an integer  $m$  such that for an integer  $n$   $180^\circ \times m < \arg(z^n) < 180^\circ \times (m+1)$  holds. We can find it by checking how the sign of  $\text{Im } z^i$  changes for all integers  $i$  such that  $i < n$ . For example for  $z=2+3i$ ,  $\text{Im } z^3 = 9$  and  $\text{Im } z^4 = -120$ . So we can see that  $\arg(z^3) < 180^\circ < \arg(z^4) < 360^\circ$ . If  $\arg(z)$  is satisfied by the inequality  $180m < \arg(z^n) < 180(m+1)$ , then  $\arg(z)$  is between  $\frac{180m}{n}$  and  $\frac{180(m+1)}{n}$ . If  $n$  is large enough, the approximate value of  $\arg(z)$  is  $\frac{180m}{n}$ .

However, it is difficult to calculate the value of  $\text{Im } z^i$  for large  $i$ .

#### 5. PC-Arg : a program for computing the value of an argument

I also wrote the Visual Basic program named "PC-Arg" to obtains the value of  $\arg(a+bi)$ ,  $a$  and  $b$  integers. The involutions of  $z=2+3i$  and their values are shown in figure 4. At the same time, the argument for  $z^n$  is indicated every  $180^\circ$ . If we want to see the lower part, we can do it by moving the vertical scroll bar. If the absolute value of  $\text{Re } z$  or  $\text{Im } z$  becomes too large to display in a single line, it is displayed in two lines. If it grows still more, the right part of the frame can be seen by moving the horizontal bar.

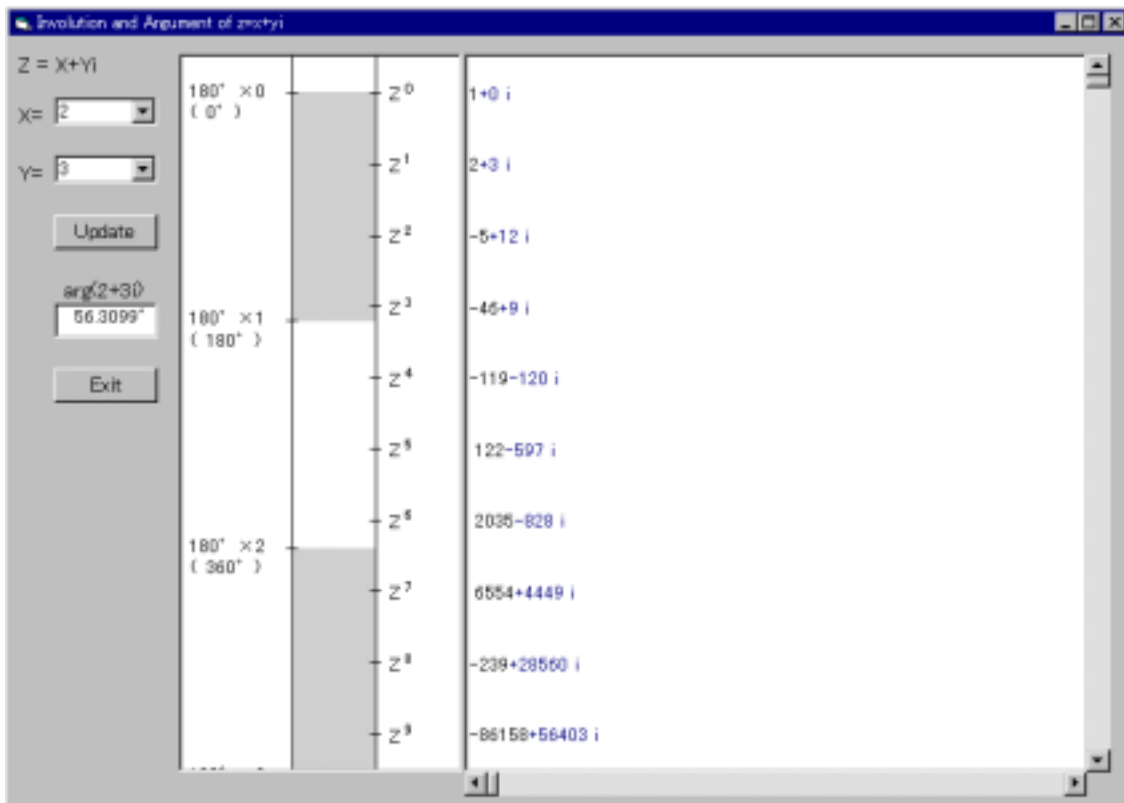


Fig 4 Main window of the PC-Arg program which computes the value of an argument

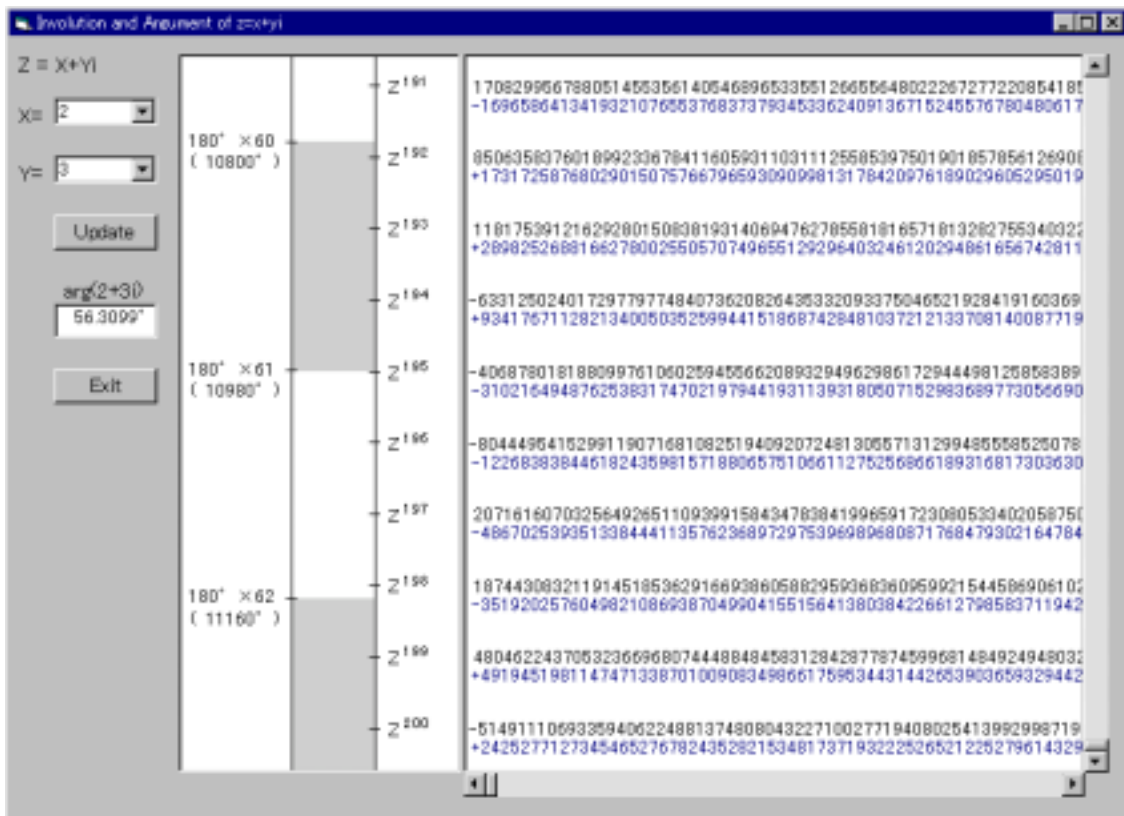


Fig 5 Example showing that  $\arg((2+3i)^{195})$  is close to  $10980^\circ$

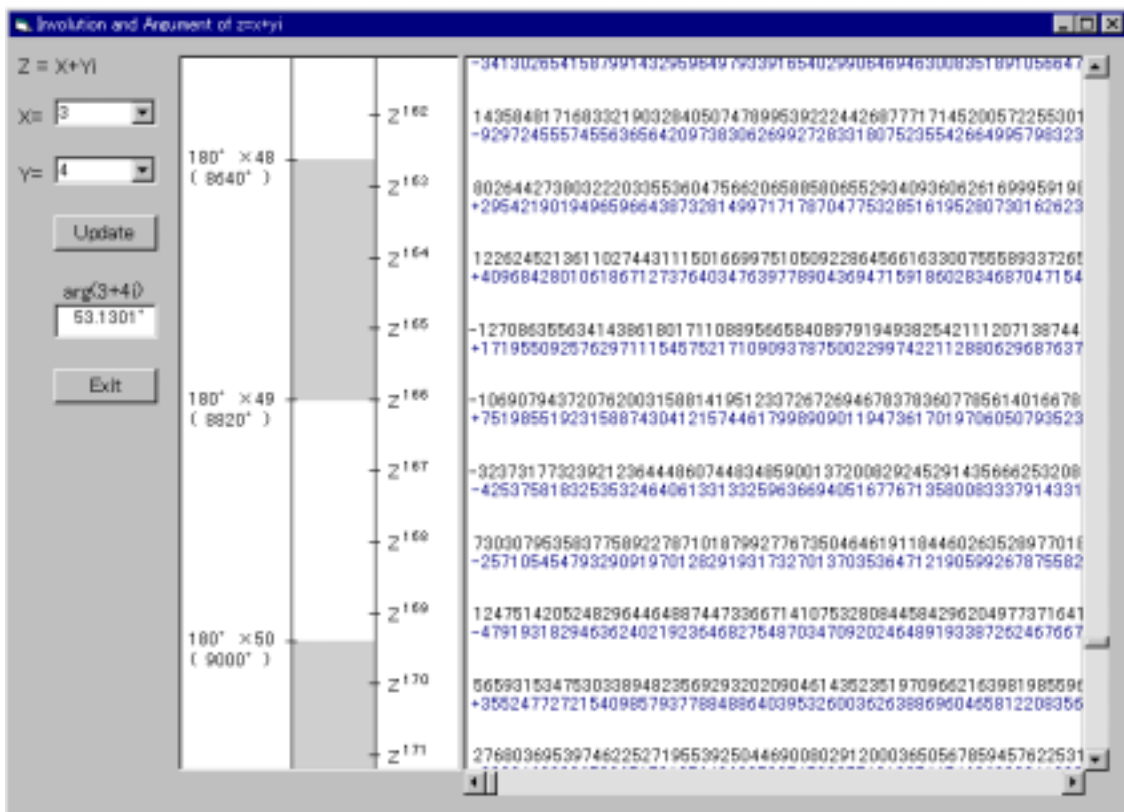


Fig 6 Example showing that  $\arg((3+4i)^{186})$  is close to  $8820^\circ$

In figure 5,  $\arg(z^{195})$  nearly equals  $10980^\circ$ . So, we can see that the approximate value of  $\arg(z)$  is  $56.308^\circ (=10980/195)$ , the true value being  $56.3099^\circ$ . If we can compute the  $n$ th power of  $z$  for a bigger  $n$ , we can increase the accuracy of the value.

Of course, we can compute other Gauss Integer arguments by changing  $\text{Re } z$  and  $\text{Im } z$  by clicking on the list boxes. Figure 6 shows the example  $z = 3 + 4i$ . We can see that  $z^{166}$  nearly equals  $8820^\circ$ . So, the approximate value of  $\arg(z)$  is  $53.133^\circ (=8820/166)$ , the true value being  $53.1301^\circ$ .

## 6. Structure of the logarithm and argument

We have used the idea of involution to obtain the value of a logarithm and an argument. During the process, we found a common structure in both. In the case of the logarithm, we compute the involution of its base; in the case of the argument, we compute the involution of  $z$ . In the case of the logarithm, we want to investigate what is the power of the base for the antilogarithm involution; in the case of the argument, we check how many times the involution of  $z$  rotates around the origin on the complex plane.

In other words, in the case of the logarithm, the value of  $\log_a m$  equals the slope of the graph shown in figure 7 where the horizontal axis is the exponent of  $m$  and the vertical axis is the exponent of  $a$ ; in the case of the argument, the value of  $\arg(z)$  equals the slope of the graph shown in figure 8 where the horizontal axis is the exponent of  $z$  and the vertical axis is the argument.

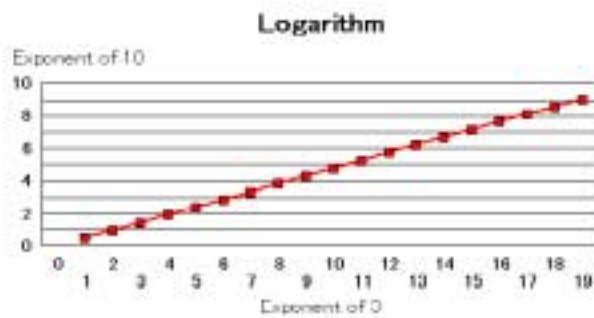


Fig 7 Logarithm example ( $\log_{10} 3$ )

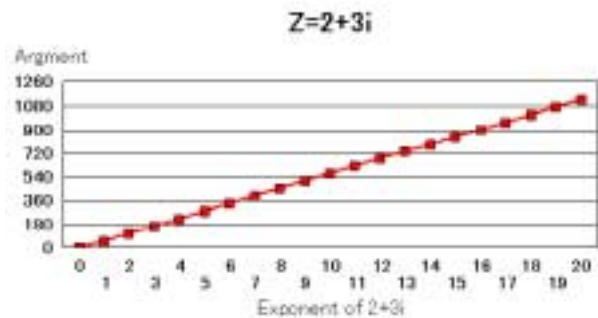


Fig 8 Arg(2+3i) example

Both methods are similar as can be seen from the following formulae.

$$\log_a m^n = n \log_a m \quad \arg(z^n) = n \arg(z)$$

Recalling the following formula for logarithms.

If the natural number  $N$  is factorized into  $p^l q^m r^n$  ( $p$ ,  $q$  and  $r$  prime numbers,  $l$ ,  $m$  and  $n$  natural numbers), then  $\log_a N = l \log_a p + m \log_a q + n \log_a r$

Then, does a similar formula exist for the argument of a Gauss Integer? If Gauss Integer  $z$  is

factorized into two Gauss Integers (i.e.  $z = v \times w$ ), then  $\arg(z) = \arg(v) + \arg(w)$ . Moreover, if  $v$  is factorized into  $v_1 \times v_2$  in a Gauss Integer, then  $\arg(z) = \arg(v_1) + \arg(v_2) + \arg(w)$ . If  $v_1$ ,  $v_2$  and  $w$  are irreducible elements in a Gauss Integer, then we cannot factorize  $\arg(z)$  anymore. Thus, each  $\arg(z)$  is factorized into the sum of arguments of some irreducible elements.

### 7. PF-Gauss : a program for factorizing a Gauss Integer

Finally, I wrote the Visual Basic program named “PF-Gauss” which factorizes a Gauss Integer into the product of some irreducible elements through trial and error.

The main window of the PF-Gauss is shown in figure 9. The window contains the complex plane and some buttons. The Gauss Integer  $a + bi$  surrounded by a rectangle is the number that we will factorize into the product of some irreducible elements. In other words, we will check whether this Gauss Integer can be divided by other Gauss Integers. (i.e. the quotient is also a Gauss Integer.) We should note that each Gauss Integer  $z$  can always be divided by  $1$ ,  $-1$ ,  $i$ ,  $-i$ ,  $z$ ,  $-z$ ,  $iz$  and  $-iz$ , so division by these numbers can be excluded. The question is whether  $z$  can be divided by other Gauss Integers. In order to find the Gauss Integers which divide  $z$ , the user clicks anywhere on the complex plane. If  $z$  can be divided by the selected number, it is factorized into two Gauss Integers which are colored dark gray on the screen and the color of  $z$  returns to light

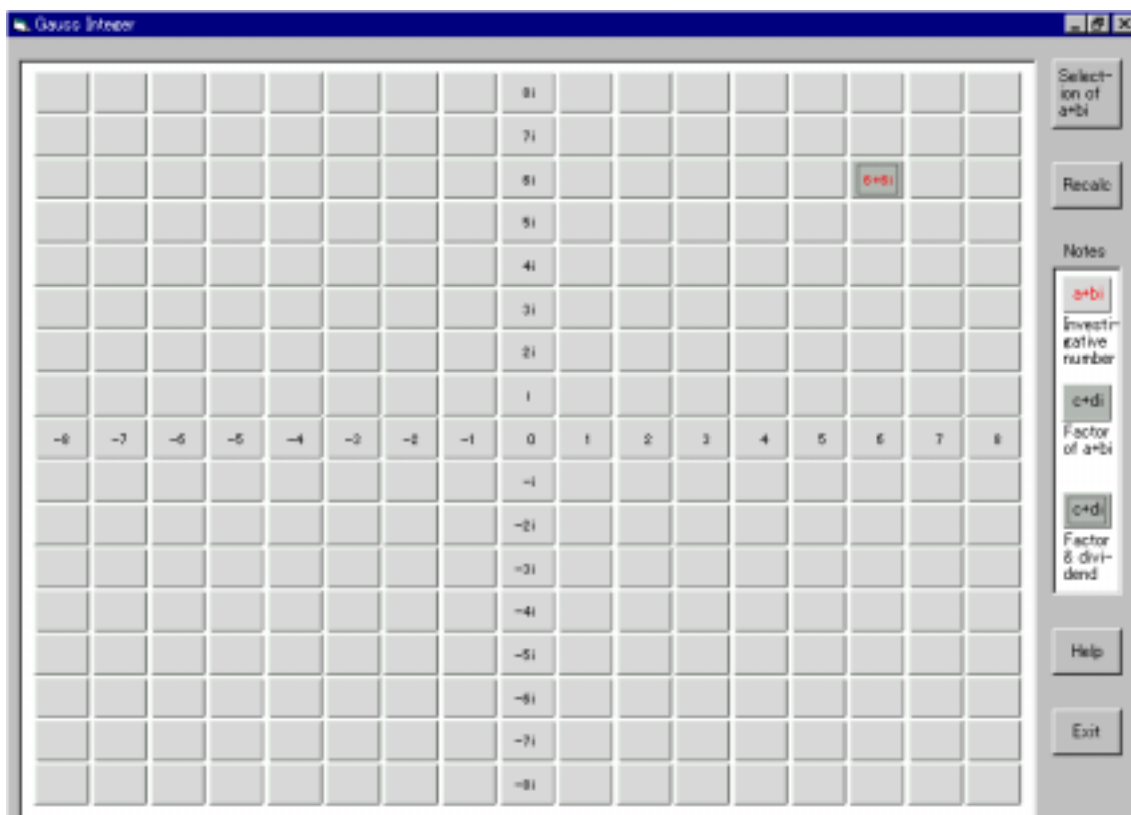


Fig 9 Main window of the PF-Gauss program which factorizes a Gauss Integer

gray. Otherwise, the screen does not change.

In the case shown in figure 9,  $z = 6 + 6i$  is surrounded by a rectangle. If we click the number 3, then  $6 + 6i$  is factorized into 3 and  $2 + 2i$ . (See figure 10.) We can see that the color of 3 and  $2 + 2i$  becomes dark gray and the color of  $6 + 6i$  returns to light gray. Moreover,  $2 + 2i$  is surrounded by a rectangle. It means that  $2 + 2i$  is a dividend. In order to factorize  $6 + 6i$  into some irreducible elements, it is necessary to factorize each factor of  $6 + 6i$  into smaller ones. We should note that the product of the numbers which are colored dark gray always equals  $6 + 6i$ . If we click the number 2, then  $2 + 2i$  is factorized into 2 and  $1 + i$ . The color of  $2 + 2i$  returns to light gray and the color of 2 and  $1 + i$  becomes dark. (See figure 11.) We can see that the number



**Fig 10**  $6 + 6i = 3 \times (2 + 2i)$  example



**Fig 11**  $6 + 6i = 3 \times 2 \times (1 + i)$  example

$1 + i$  is surround by a rectangle. But we cannot factorize it anymore into two Gauss Integers of which its absolute value is either 1 or its absolute value. This means that the number  $1 + i$  is an irreducible element. Then we have to change the dividend of the number  $1 + i$  to 2. To do this, the mouse is moved to the number 2 and the right button clicked. The number 2 is surrounded by a rectangle, meaning that the number 2 becomes a dividend. The rest of the process is omitted for brevity.

Finally,  $6 + 6i$  is factorized into  $3(1 + i)^2(1 - i)$ . Each factor is an irreducible element. ( See figure 12) If we use the units  $(1, -1, i, -i)$ , then each factor can be expressed as factor of quadrant 1 and some units. In this case,  $1 - i$  is associated with  $1 + i$  because  $1 - i = (1 + i)i^3$ . Then



**Fig 12**  $6 + 6i = 3 \times (1 + i)^2 \times (1 - i)$  example



**Fig 13**  $6 + 6i = 3 \times (1 + i)^2 \times i^3$  example



$6 + 6i = 3(1 + i)^3 i^3$ . If you want to factorize  $6 + 6i$  into these factors on the screen, move the mouse to the number  $i$  and click three times in order to divide  $1 - i$  by  $i^3$ . Then we can see that  $6 + 6i$  is factorized into  $3(1 + i)^3 i^3$  on the screen. (See figure 13.) In order to investigate another Gauss Integer, click the upper right button labeled “Selection of  $a + bi$ ” and then, click the desired Gauss Integer.

In the case of  $z = 6 + 7i$ , it is factorized into  $1 + 2i$ ,  $1 + 4i$  and  $i^3$  by the PF-Gauss and  $1 + 2i$  and  $1 + 4i$  are irreducible elements. Then,  $\arg(6 + 7i) = \arg(1 + 2i) + \arg(1 + 4i) + 270^\circ$  (See figure 14.)

In addition, the irreducible elements of the Gauss Integers can be optionally displayed on the complex plane as shown in figure 15.



Fig 14  $6 + 7i = (1 + 2i) \times (1 + 4i) \times i^3$  example

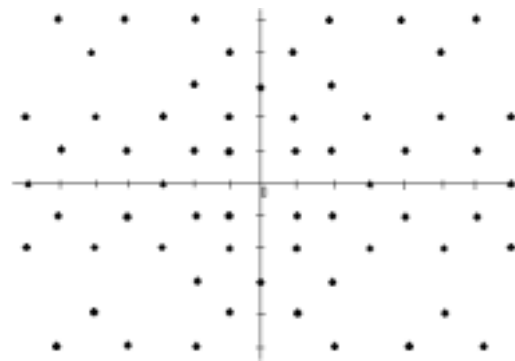


Fig 15 Distribution of irreducible elements

## 8. Discussion

I taught the students (12th grader) how to compute the logarithm value and the value of the argument of a Gauss Integer using the programs PC-Log and PC-Arg, the next step being the factorization of a Gauss Integer. They showed interest in the computation method as well as the in the way the computer displays the value of the involutions. Following is a summary of their impressions:

- Before, I had to use a number table for the logarithm or a trigonometric function in order to obtain the values. But now I can compute them. It is wonderful!
- I was surprised to know that we can obtain the logarithm value or the value of the argument by investigating large numbers.
- I thought it was impossible to compute the logarithm value or the value of the argument without using high level mathematics. But now I can do it myself using this simple idea.
- It was a useful lesson for me. I think mathematics is profound. It was a wonderful experience.
- It was a delightful time for me. Now, I am more and more interested in mathematics.
- I was surprised to find out that we can obtain the logarithm value and the value of the argument

the same way. The idea of involution is wonderful!

- I have learned a wonderful method. It makes the impossible possible. This method expands my mathematical world.
- I have learned new exponent numerals both in Japanese and English as well as a method to obtain the logarithm value and the value of the argument. It was a fruitful time!

The final objective for the author was not to solve a specific mathematical problem but, as stated above to expand the mathematical understanding of the students and to give them specific examples of the usefulness of the computer.

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