

The Algebraic Calculator and Mathematics Education

Barry Kissane

The Australian Institute of Education
Murdoch University, Murdoch, Australia 6150
kissane@murdoch.edu.au

Abstract

This paper describes the recent development of hand-held algebraic calculators and evaluates their significance for secondary education. Sophisticated computer algebra systems (CAS) have been available to mathematicians for some years now but have been too powerful, too sophisticated and too expensive and have required too much access to powerful computers to have had much impact on teaching and learning elementary algebra and calculus. Unlike CAS, algebraic calculators have been developed to meet the needs mathematics students rather than those of mathematicians, scientists and engineers. Access to an algebraic calculator will allow students to deal with all of the symbolic manipulation demands of the conventional secondary school algebra and calculus curriculum. The implications of such access for the mathematics curriculum, teaching methods, assessment and the professional development of teachers are discussed in the paper. Alternative ways of regulating access to and controlling the use of algebraic calculators are discussed, together with the desirability of doing so. Parallels are drawn between the significance of the algebraic calculator for the secondary school and of the arithmetic calculator for the elementary school.

Introduction

The idea of using computer technology for symbolic manipulation purposes is not a new one. Useful and powerful software that could deal with the more routine aspects of algebra and calculus first appeared in the 1970's on mainframe computers and were available on microcomputer platforms by the late 1970's and early 1980's. Since then, such software has developed considerably in sophistication, usability and the range of capabilities. Indeed, from the outset of the Asian Technology Conference on Mathematics four years ago, computer algebra systems (CAS) such as *Maple* and *Mathematica* (among others) have featured prominently on the conference programs. For the most part, interest in CAS has been strongest among research mathematicians and senior undergraduates, as well as computer scientists interested in how such software is optimally designed.

The same conference programs have included papers concerned with graphics calculators, arguably of increasing importance to mathematics education because of their potential accessibility to secondary school students as well as early undergraduates. (Kissane, 1995). It is now often stated that a graphics calculator is also a computer, albeit a relatively small, inexpensive and limited one. The present paper is an attempt to summarize how much the gap has closed between CAS and graphics calculators with the development of algebraic

calculators and to consider the implications of this development for mathematics education in secondary schools and the early undergraduate years.

The Progression of Calculators

Although there are variations on the themes, it is now possible to identify four distinctly different levels of calculators, all of which have been developed (or at least refined) with mathematics education in mind. Calculators at each level generally have the capabilities of calculators at previous levels.

Arithmetic calculators

Arithmetic calculators (sometimes called four-function calculators) allow for arithmetic computation to be carried out with small numbers. They meet the needs of *everyday calculation* for most people and thus most of the computational needs of elementary school students. These are in widespread use almost everywhere in society (except some school systems, paradoxically), especially in business and commerce. They are typically described as providing one of three ways of computation (mental, calculator and paper) by many school curricula.

Scientific calculators

Scientific calculators extend arithmetic calculators by providing access to mathematical tables (such as trigonometric and logarithmic tables), some statistical computation. They also handle larger and smaller numbers, using scientific notation. They deal with most aspects of *scientific calculation* and are in use in very many secondary school systems around the world. Recent models have been tailored more clearly to the needs of secondary school students rather than scientists and engineers.

Graphics calculators

Graphics calculators have many more capabilities than scientific calculators, typically allowing lists and complex numbers to be dealt with and include many more computational capabilities (such as numerical differentiation and integration, equation solving, matrix arithmetic, recursion). The graphics screen (after which such calculators are named) provides opportunities for function and statistical graphing. Significant memory storage means that data can be stored and analyzed and elementary programming is possible. Graphics calculators permit *quantitative exploration* by students. All models have been developed with the field of education in mind as the major market.

Algebraic calculators

Algebraic calculators include the symbolic manipulation capabilities characteristic of secondary school algebra and calculus, such as: expanding, factorizing and simplifying expressions (both algebraic and trigonometric); substitution of variables; solving equations, inequalities and systems of equations; differentiating and integrating elementary functions; finding sums of series; and evaluating limits. Some models include other capabilities such as producing Taylor Series expansions, solving differential equations and manipulating matrices which include symbolic expressions. Some examples of these capabilities are provided in the next section. Algebraic calculators permit *mathematical exploration* by students.

Algebraic Capabilities

To illustrate typical capabilities of algebraic calculators, a number of examples have been chosen, using the Casio Algebra FX 2.0 calculator, released to schools during 1999. While this particular calculator is not the most powerful available, in terms of mathematical capabilities, it has been designed with the needs of unsophisticated

students in mind. (For example, it includes a tutoring aspect, designed to help students see how to solve various kinds of linear and quadratic equations.)

In devising suitable learning activities for senior secondary school students, Etchells *et al* (1997) provided examples of typical CAS operations available on various platforms: *approximate*, *expand*, *factorize*, *simplify*, *substitute*, *differentiate*, *integrate*, *solve*, *limit* and *sum*, together with some graphing commands such as *plot*, *scale* and *zoom*. The screens below have been chosen to illustrate some of these various generic capabilities as they appear on the Algebra FX 2.0.

Figure 1 shows two of the basic algebraic commands, factorizing and expanding. Although the calculator entry syntax is a little awkward (for example, using ^ for exponentiation and upper case letters throughout), the results are given in slightly more conventional algebraic notation.

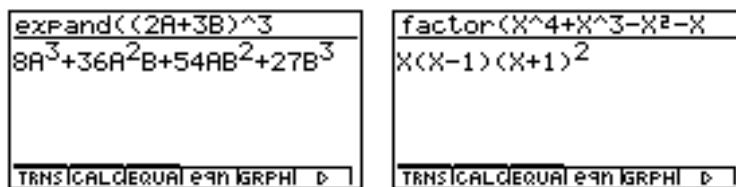


Figure 1: Expanding and factorizing elementary expressions

Figure 2 shows that commands can be combined together to make more complicated commands. In this case, the calculator is finding the sum of the squares of the first N integers, giving the result in factorized form.

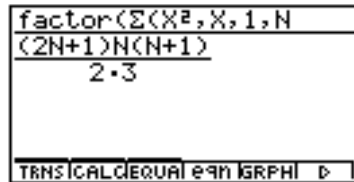


Figure 2: Combining commands (factorization and summation)

Equations can be solved (symbolically) with a single command, as shown in Figure 3, although not all elementary equations have a closed form solution.

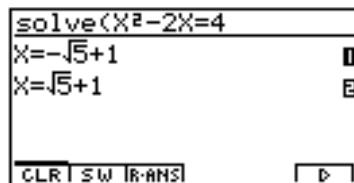


Figure 3: Solving exactly a quadratic equation

One of the features of a calculator designed for educational use is that students might use it to see the steps along the path to a solution, if desired. To illustrate this idea, the calculator screens in Figure 4 show one possible set of steps carried out successively to solve the same equation as that shown in Figure 3. The final screen shows that the whole series of steps can be recalled, so that students can see where they have been in seeking a solution.

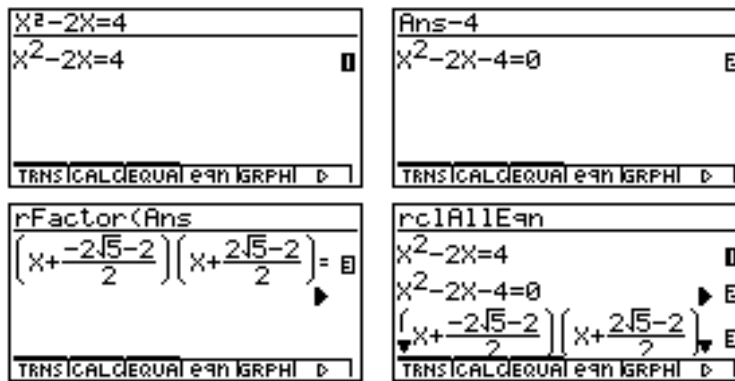


Figure 4: Steps in solving a quadratic equation

Similar kinds of things are possible with the solution of inequalities, as suggested by Figure 5, which shows only the short and simplified version of the solution, using conventional notation for an interval.

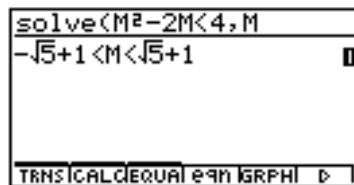


Figure 5: Solving a quadratic inequality

Figure 6 shows two examples of symbolic manipulation in elementary calculus, one concerned with differentiation and the other with finding an infinite limit.

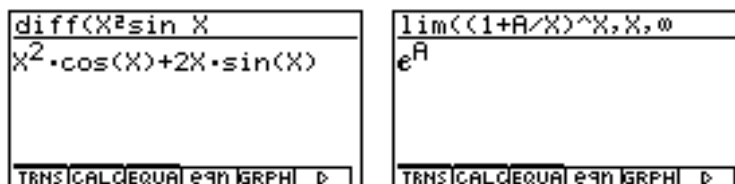


Figure 6: Elementary calculus operations

Finally, Figure 7 shows examples of integration on the calculator. Both indefinite integrals and definite integrals are available, with results given exactly in the latter case.

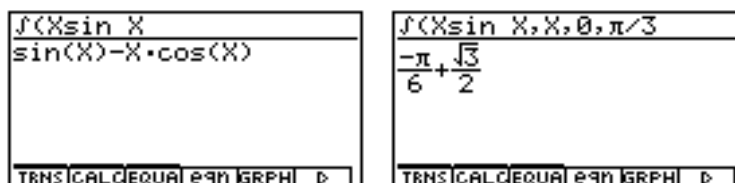


Figure 7: Indefinite and definite integration

These examples together suggest that much, if not all, of the symbolic manipulation demands of conventional secondary school mathematics can be completed on an algebraic calculator

A Continuum of Responses

In considering how we might respond to this technological development, Kissane, Bradley & Kemp (1996) suggested that it might be helpful to think about algebraic calculators in secondary mathematics education in the light of four-function calculators in elementary school arithmetic, despite the hazards of reasoning by analogy. They identified the following continuum of responses to the arithmetic calculator, in a sequence from tighter to looser control by the teacher or other educational authorities.

Prohibition

For some elementary school students, calculators are still prohibited (in school). In some cases, it is a general prohibition, while in other cases, it is more particular. (For example, they cannot be used during assessment.) To date, using technology for symbolic manipulation has been prohibited in most schools, partly because it is too expensive. Equity issues associated with examinations are obvious if only some students have access (although it is interesting to note that the Advanced Placement calculus examinations in the USA now permit at least three different algebraic calculator models for examination use.) Prohibition is a risky strategy, as French (1998, p.70) notes:

We may just ignore such developments in the hope that they will go away, in which case many students are likely to become machine dependent or be put off mathematics altogether because readily available technology is ignored.

An additional problem with this strategy is, of course, that we are unlikely to learn anything about the matter by a policy of prohibition.

Checking

Elementary school students might be allowed, or even encouraged, to use their calculator to check their arithmetic. It is still expected that they will do their work without the calculator first, and they may even be denied access to a calculator until quite late in their school career. Although this practice is hard to defend, it still seems quite common. It is conceivable that similar uses for an algebraic calculator might be contemplated, with algebra and calculus instead of arithmetic. If the use of algebraic calculators is restricted to checking, students (and others) will realize that this practice will not deal with the essential issue of why it is necessary to learn to do by hand what a machine does more efficiently, reliably and quickly. Such a reaction would reflect that of many people restricted to using arithmetic calculators in such a restricted way.

Substitution

Usually (but not always) without sanction from their teacher, elementary school students might use their calculators to do arithmetic *instead* of learning to do so mentally or with paper and pencil. The analogy with a symbolic manipulation device is easy to make. At the least, substitution is rendered possible by technology. If we want to prohibit students from doing this, we need to be able to defend our policy. To dissuade students from substituting inappropriately with algebraic calculators, we will need better arguments than merely, "the batteries might go flat" or "you won't really understand what you are doing unless you do it the long way by hand".

Simultaneous use

Developing arithmetic competence may take place in an environment in which paper-and-pencil, mental and calculator work all happen together. At issue is the locus of control: whether it is the teacher or the students who decide which kinds of technologies to use at a particular time. It seems likely that this sort of environment is the most likely one for students to develop some discretion about when to use a calculator and when not to use a calculator, although some explicit attention needs to be paid to helping them make such decisions. An algebraic calculator such as the Casio Algebra FX 2.0 will provide an expectation that students make their own decisions about what to do. For example, Figure 8 shows the transformation menu, which makes clear that a number of equivalence transformations are available for dealing with symbolic objects

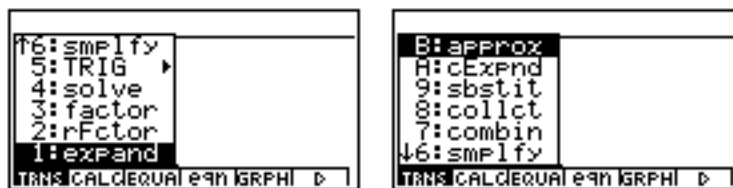


Figure 8: Equivalence transformations available

Students must learn what these are, as well as why, where and when to use them. Access to an algebraic calculator may help this sort of learning by focussing on these contextual questions rather than on the mechanics of performing the transformations by hand.

Complexity

Elementary school students may be encouraged to use their calculators for complicated situations, such as those involving large numbers, those for which numbers are not integers or those requiring many successive calculations. In the analogous way, an algebraic calculator might be used when a situation demands particularly complicated algebraic manipulations or especially difficult integrals, for which general solutions are sought.

This continuum of responses is suggested as a first step in considering these new forms of technology from an educational perspective. Now that we have a generation of experience with less sophisticated technology, we may be able to learn something from it when thinking about more powerful technologies.

Symbol Sense

In the same way that the development of less sophisticated calculators has generated interest in 'number sense' in recent years, it now seems important to consider the analogous situation for symbolic manipulation associated with secondary school algebra and calculus. The best formulation of this has come from Arcavi (1994), who described *symbol sense* as:

... a complex and multifaceted "feel" for symbols. Paraphrasing one of the definitions provided by the Oxford Encyclopedic English Dictionary for the word "sense", symbol sense would be a quick or accurate appreciation, understanding or instinct regarding symbols. (p. 31)

In his seminal paper, Arcavi (1994) suggested that symbol sense includes (but is not restricted to) the following aspects:

- An understanding of and aesthetic feel for the power of symbols
- A feeling for when to abandon symbols in favor of other approaches
- An ability to manipulate and to "read" symbolic expressions as two complimentary aspects of solving algebraic problems
- The awareness that one can engineer symbolic relationships and the ability to do so
- The ability to select a possible symbolic representation of a problem, to acknowledge dissatisfaction with a choice and to be resourceful in finding a better replacement
- The realization of the constant need to monitor and compare the meanings of symbols with one's intuitions when solving a problem
- Sensing the different roles symbols can play in different contexts (p.31)

Similarly, French (1998) referred to 'mental algebra' as a parallel idea to mental arithmetic, and similarly important for senior secondary school mathematics in a technological age:

Students need an understanding, knowledge and certain skills that they have 'at their fingertips' in the sense that they can immediately call to mind particular key ideas, explain them simply and do simple calculations with them, without reference to text or machine, and without extensive written working. (p.66)

This kind of thinking goes to the heart of what is important about secondary school algebra (and, to a lesser extent, calculus). For generations, the focus for many students in secondary school has been the development of procedural skills with algebraic expressions, often in quite complicated situations. Many students have required a great deal of time to develop such skills, while many others have abandoned hope of doing so relatively early. Although we have long realized that competence with the skills themselves, while necessary to making progress in mathematics, does not necessarily reflect a sound understanding of algebra (or calculus), our common practices do not seem to reflect this.

For example, formal assessment (the most powerful way in which we communicate our goals and what we value) frequently includes symbolic manipulation in both algebra and calculus, apparently for its own sake. Tasks that begin with imperatives such as 'expand', 'factorize', 'simplify', 'solve', 'differentiate' or 'integrate' (or their symbolic abbreviations) can usually be answered by the routine application of symbolic skills. Some of these are rather complicated (such as integration by parts or partial fractions), but nonetheless they still demand only procedural skills from students. Thus, almost a decade ago, Bibby (1991) noted:

For many students current practice in A-level mathematics seems largely to consist of the assimilation, rehearsal and implementation in stereotyped contexts of a more-or-less well-defined set of standard algorithms—in short, "plug-and-chug" mathematics, as Philip Davis has described it. With the aid of computer algebra systems, demonstrations of "A-level papers in ten minutes" have recently been possible, and this clearly illustrates the essentially "plug-and-chug" nature of the assessment tasks. (p. 40)

While computer algebra systems have been confined to desktop computers and priced beyond the means of the great majority of students, it has been possible (although clearly undesirable) to choose not to respond to this situation. However,

the development of hand-held, portable and relatively inexpensive versions of CAS in the form of algebraic calculators gives rise to a new imperative to reconsider the matter.

Some Educational Directions

In considering possible educational directions associated with algebraic calculators, it is interesting to continue the process of reasoning by analogy. The Calculator Aware Number project, directed by the late Hilary Shuard in the UK in the 1980's involved allowing children just entering elementary school unrestricted access to arithmetic calculators. Contrary to the expectations of some, the longitudinal project found that students used the calculators to help them learn about number, and did not become dependent on them for calculation. Indeed, it was reported that many students developed a culture that prized mental arithmetic and their own ways of calculating, rather than using the calculator as a crutch. Torres-Skoumal (1999) reported informally on the equivalent kind of experience with 9th-grade Austrian students learning algebra with regular access to the powerful Texas Instruments TI-92 algebraic calculator:

Since the machine renders all solutions exact (unless specifically instructed to do otherwise) the students have developed a natural preference for fractional, surd or transcendental answers over decimal, approximate answers. It is ironic indeed that this latest stage of technology is bringing back the "beautiful numbers" whose loss was one of the greatest criticisms aimed at all previous generations of calculators. Make no mistake; a machine with CAS is a mathematician's tool. Just as with numbers, the machine that can do algebra for my students has actually made them better at algebra.

Also in Austria, Kutzler (1999) has suggested that one of the plausible reasons for the value of an algebraic calculator to young students is that it permits students to attend to higher-order processes (such as deciding what operation to perform next) rather than becoming distracted by lower-order operations (such as carrying out a particular equivalence transformation). He suggests that this kind of scaffolding is useful to students even if the ultimate goal is to develop traditional symbolic manipulation skills in a context in which neither the curriculum nor the associated means of assessment are changed.

It seems increasingly difficult to argue, however, that neither the curriculum nor the assessment procedures associated with elementary algebra should remain immune to influence from technology of this kind. At the very least, some reconsideration of the balance of procedural skill, conceptual content and strategic thinking associated with algebra and calculus seems necessary. In this vein, both Heid *et al* (1995) and Etchells *et al* (1997) have provided valuable collections of ideas and activities that incorporate symbolic manipulation tools using technology. Similarly, Kissane, Bradley & Kemp (1996) give examples of activities that use symbolic manipulation on a calculator to help students learn important mathematical ideas in both algebra and calculus. The great majority of these kinds of activities can be completed using an algebraic calculator designed for school use, rather than a more sophisticated CAS, devised mainly for professional use. These rich collections each serve to show how access to technology can be a source of mathematical enrichment and insight, likely to aid learning of important material and to provide students with access to new ways of solving problems.

Bibby (1991) noted that two concerns likely to be prominent in curriculum thinking associated with algebraic calculators are those of calculator abuse and calculator dependence. Indeed, such concerns have been frequently voiced before in

the context of less sophisticated calculators. It is important to acknowledge the concerns: no-one is likely to be comfortable with the idea of students using algebraic calculators inappropriately (such as to factorize $x^2 + x$ or to solve $3x - 2 = 7$); nor are we likely to be comfortable with students unable to do any symbolic manipulation without their calculator. Avoiding such problems would seem to be a matter of conscious planning rather than pious hope. For example, students who are denied access to calculators may well come to see their occasional use as opportunities to avoid thinking. In addition, it is much too optimistic to expect students to learn discriminating use of algebraic calculators unaided. Only if the experience of using calculators is part of the curriculum, and thus part of the work and responsibility of the classroom teacher, can we expect that students will get real help in learning how to avoid dependence.

Change in education is always a difficult matter, especially so for classroom teachers of mathematics, whose energies and intellectual resources are mostly consumed by the daily realities of teaching. In many countries, teachers have yet to come to terms with the implications of the graphics calculator, providing access to quantitative exploration as suggested above. In some cases, curricula and assessment methods have adjusted to this form of technology, but there are many others for which this is not yet so. Many teachers will find the prospect of coming to terms with the much more far-reaching changes associated with access to algebraic calculators daunting, to say the least. (Indeed, the first reaction of many teachers to this sort of technology is a mixture of apprehension and awe, particularly those who lack confidence in their own command of mathematics.) It is crucial that teachers are provided with ample support and reasonable time frames for any changes. Although there will always be teachers eager for new challenges, happy to take the lead in adapting curriculum and teaching to new technologies, there will normally be many more for whom such an expectation is quite unreasonable. Curriculum change that does not deal with this reality is unlikely to lead to real change and certainly unlikely to be successful.

By way of example, a recent paper by a self-confessed enthusiast for new technology (Podlesni 1999) expressed concern with the rate of change of technology for mathematics education. Indeed, he questioned the source of the changes:

Are we getting to the point where technology companies are making de facto curriculum decisions for us? Are they paving the way, consciously or unconsciously, for their future leadership in that process by making calculators upgradeable—through their software, one presumes? ... Are we doing our job as teachers or relinquishing part of it to the electronics industry? Are we becoming unpaid salespeople for that industry with every new model? (p. 89)

Podlesni's concerns are understandable, although the necessity of education responding to the changing world external to the school is neither new nor inappropriate. As for other changes such as television and educational computing, it is less important who is asking the critical educational questions than it is that someone is trying to answer them. A healthy dialogue between educators and industry people is clearly desirable, so that we understand each other and can learn from each other. But while it is important to keep the needs of classroom teachers in mind when technological change is underway, it is also important that the messenger not be shot in the process. In the case of algebraic calculators, an important part of the message is that it is now possible to manufacture relatively inexpensive and powerful hand-held technologies that at first glance seem to be capable of performing the symbolic rituals of secondary school algebra and

calculus. It is important to know this as soon as possible in order to give ourselves the greatest chance of exploiting it for educational gain and adapting our conventional practices sensibly to it.

Clearly, more research and more time are needed before we will have good answers to the critical questions of which activities and which kinds of uses for algebraic calculators are most beneficial, part of the process of providing guidance and support to classroom teachers as well as curriculum developers). To date, the limited experiences reported suggest that the technology is more likely to be helpful rather than harmful to students.

Conclusion

Symbolic manipulation on hand-held calculators which are affordable to many students and schools is already a reality. The major implications for practice may be a consequence of providing students with access to opportunities for mathematical exploration that would not otherwise be available to them. Inevitably, the development of algebraic calculators will demand that we look more carefully than before at what is crucially important about algebra, how to develop appropriate symbol sense in students and what can be comfortably left to a machine. It is unlikely that strategies based on ignoring or marginalising such technologies will provide much insight into how to deal with them. The real needs of teachers must be adequately taken into account, if genuine progress is to be made.

References

- Arcavi, A. 1994. Symbol sense: Informal sense-making in formal mathematics, *For the Learning of Mathematics*, 14, 3, 24-35.
- Bibby, N. 1991. Wherefore "plug-and-chug"?: Computer algebra versus A-level mathematics, *Mathematical Gazette*, 75, 40-48.
- Etchells, T., Hunter, M., Monaghan, J., Pozzi, S. & Rothery, A. 1997. *Mathematical Activities with Computer Algebra*, Bromley, UK: Chartwell-Bratt.
- French, D. 1998. Matters for debate: School algebra and the challenge of the TI-92, *Mathematical Gazette*, 82, 493, 62-71.
- Heid, K., Choate, J., Sheets, C. & Zbiek, R. 1995. *Algebra in a Technological World*, Reston, VA: National Council of Teachers of Mathematics.
- Kissane, B., Bradley, J. & Kemp, M. 1997, Symbolic manipulation on a TI-92: New threat or hidden treasures? In N. Scott & H. Hollingsworth (Eds), *Mathematics: Creating the future*, Melbourne, Australian Association of Mathematics Teachers, 388-396.
[<http://www.edfac.unimelb.edu.au/DSME/TAME/AAMT97/papers/symbmani/index.htm>]
- Kissane, B. 1995. The importance of being accessible: The graphics calculator in mathematics education. In *First Asian Conference on Technology in mathematics*: 161-170. Singapore: The Association of Mathematics Educators.
- Kutzler, B. 1999. *The Algebraic Calculator as a Pedagogical Tool for Teaching Mathematics*, <http://www.kutzler.com/bk/a-pt/ped-tool.htm>. (June 13)
- Podlesni, J. 1999. A new breed of calculators: Do they change the way we teach?, *Mathematics Teacher*, 92, 2, 88-89.
- Shuard, H. et al 1991, *Calculators, Children and Mathematics*, London, Simon & Schuster.
- Torres-Skoumal, M. 1999. *Calculator Wars Part 4: The Phantom Math-Menace*. <http://www.kutzler.com/bk/art-marlene.html>. (June 13)