Triangular Squares using Mathematica

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Abstract. This note concerns generating perfect square triangular numbers using a short program of MATHEMATICA which makes use of our main theorem. All the triangular numbers are given by $t_n = \frac{n(n+1)}{2}$ for all positive integer n. We have computed the first forty-nine integers that are simultaneously triangular and square.

Triangular numbers are the set of numbers which represent the number of objects which can be used to build a triangle. The numbers 1,3,6,10, 15,⋯⋯⋯are the set of triangular numbers. All the triangular numbers can be written as $t_n = \frac{n(n+1)}{2}$ for all positive integer n , see [1] for more discussion. Square numbers are the set of numbers which represent the number objects which can be used to build a square. These are more familiar to us since they represent the "squares" of the integers. The numbers 1,4,9,16,25,⋯⋯..are the set of square numbers. In this note we would like to find all the triangular numbers that are square numbers. That is , find m such that $t_n = m^2$ for some positive integer m. That is to solve the equation $\frac{n(n+1)}{2} = m^2$. The solution of the equation is given by the quadratic formula, $n = \frac{-1+\sqrt{1+8m^2}}{2}$ when $1+8m^2$ is a perfect square. That is, $1+8m^2 = k^2$, for some positive integer k. Now we need to solve $k^2 - 8m^2 = 1$ for pair of positive integers (k,m). We observe that (3,1) is the least positive solution that is, two smallest positive integers satisfy the equation. We shall now state the main theorem.

Theorem. Since (3, 1) is the least positive solution of $k^2 - 8m^2 = 1$, then all other positive solutions (k_s, m_s) can be obtained from the equation

$$k_s + m_s\sqrt{8} = (3+\sqrt{8})^s \text{..........(1.1)}$$

by setting, in turn, s = 1,2,3,$\cdots\cdots\cdots$.

The values k_s and m_s are obtained from the term $(3+\sqrt{8})^s$ in (1.1) by the binomial theorem and equating the rational parts and purely irrational parts of the resulting equation. For example the solution (k_2, m_2) can be found by taking s= 2 in (1.1). This gives $k_2 + m_2\sqrt{8} = (3+\sqrt{8})^2 = 17 + 6\sqrt{8}$ so $k_2 = 17$ and $m_2 = 6$. Using these values, a direct calculation shows that $k_2^{\ 2} - 8m_2^{\ 2} = 1$.

It is easy to show that if k_s, m_s are calculated by the equation (1.1) then $k_s^{\ 2} - 8m_s^{\ 2} = 1$. We have, from (1.1), $k_s + m_s\sqrt{8} = (3+\sqrt{8})^s$. Since the conjugate of a product is product of the conjugates , [2], this gives $k_s - m_s\sqrt{8} = (3-\sqrt{8})^s$. Now the factor

$$\begin{aligned} k_s^{\ 2} - 8m_s^{\ 2} &= (k_s + \sqrt{8}m_s)(k_s - \sqrt{8}m_s) \\ &= (3+\sqrt{8})^s(3-\sqrt{8})^s \\ &= 1. \end{aligned}$$

Thus k_s and m_s are solutions of the equation $k^2 - 8m^2 = 1$. Based on this theorem, we produce first 49 perfect square triangular numbers (due to one page restriction) using MATHEMATICA code.

References.

1. Burton, David M., History of Mathematics, Wm C. Brown Publication, Iowa, 1995.
2. Olds, C. D., Continued Fractions, MAA, 1963.

The following is the MATHEMATICA code to generate the numbers:

```
a:=  (3 +Sqrt[8])^m;
k:= Simplify[Ceiling[a/2]];
n:= (k-1)/2;
t:= n(n+1)/2;
Table[{n,t},{m,49}]
```

Following List is the first 49 perfect square triangular numbers. List is given as a pair (n,t_n) where t_n is the nth triangular number.

{{1,1},{8,36},{49,1225},{288,41616},{1681,1413721},{9800,48024900},{57121,16314
32881},{332928,55420693056},{1940449,1882672131025},{11309768,6395543176179

6},{65918161,2172602007770041},{384199200,73804512832419600},{2239277041,25
07180834294496361},{13051463048,85170343853180456676},{76069501249,2893284
510173841030625},{443365544448,98286503002057414584576},{2584123765441,333
88478175597782548449 61},{15061377048200,11342253929403040325 0144100},{877
84138523761,385302748817947393225 0054441},{511643454094368,13088951205880
8083293251706896},{2982076586042449,4446390382511295358038307980025},{173
80816062160328,151046383493325234090009219613956},{101302819786919521,513
113064839054666370227515889 4481},{590436102659356800,17430739566178526133
1787346182798400},{3441313796169221281,59213203218523083386170674950 56251
121},{20057446674355970888,201150583547316698251648507485729739716},{1169
03366249966604049,68331985202869154322174321870197548 99225},{681362750825
443653408,23212759910620780799714104585118593683 3936},{397127313870269531
6401,788550517109077855647057812675330209745 4601},{231462760813907282450 0
0,26787504821798026311200251526376108537662 2500},{134906383349641674153 60
1,909986613424023816725161494084112360070771 0401},{786292024016459316676 6
08,309127573515950117423442905473334441338685531136},{4582845760749114225
906049,105012376334080637542298071711525298819146003 48225},{2671078254047
8226038759688,356732951962358217526390000913712681543757726308516},{15568
18494821202420066520 81,121184191290867713321430302238950786426058480941 4
1321},{907380314352243226001152800,411669517436987867075336637611518961 16
7055077474496400},{5288600036631339114000264721,13984645173728500709229 30
264856774960103726678603873 6281},{30824219905435791458000435528,475066266
389332036246720953413691967474100015647842537156},{17965671939598340963 4
002348449,161382684120635607316792831134169591445183632652406075 27025},{1
047116096470464666346013655168,54822605974377173284084890490276291894615
0251002532813381696},{6103039859426804588442079582561,18623547762876175 35
585718348358052228502459017082087504745 0641},{35571123060090362864306463
840200,632652397878046190366303389536834994771889915556907218799940100},{
2073236985011153725973967034586 41,21491557980090694297098458060768809299
959232538764024564150512761},{12083710679466018727200737569116 48,73008031
892520555991098127067660268120384201640241992796231749 3776},{70429027091
784958637230458380112 49,24801239285476898342676264744943722351630669325 1
4351352615464427562 5},{41049045187124373309618201271155848,84251205538728
933809108202005740995727423891503847703996129558787747 6},{23925136841356
774399398616178892384 1,28620608643882360596754112417206994824972492441 98
30758451578953435585 61},{1394459165294282090654298769462387200,9722581818
366129709515487401649804140917905041123861016954071460931136 00},{8127503
62335212479993180645498539936 1,330281575738009586517559030531921270842 95
904647379144381798685071822303841}}