

Samples of Animations from Teaching Engineering Mathematics with Maple V R5

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Abstract

The purpose of this paper is to show how the powerful computer algebra system Maple V Release 5 can be used to explore and visualize with animation a selection of engineering mathematical topics in calculus, differential equations and linear algebra. Animated examples which will be discussed and shown are:

- Taylor polynomial in two variables, simple pendulum with numerical solution, a system of two and three spring-coupled masses, reflection and transmission of a transverse pulse at the boundary of two ropes of different linear density, the convolution product of a rectangular window with an exponential function and trajectory of the dynamical system described by the difference equation $\mathbf{X}(k+1) = A \mathbf{X}(k)$,

Introduction

At Telemark College, Department of Technology in Porsgrunn, Norway, we have during the last four years incorporated the computer algebra system Maple V as an integrated part of the learning process throughout all the subjects in the engineering mathematics curriculum. Maple's capability of both symbolic and numerical computation and graphical visualization permit us to develop greater conceptual understanding of the process under study not evident before the advent of a CAS tool like Maple. An important part of the design of a process is the modeling and simulation. The Maple system enables us to develop both simple and complex mathematical models in classrooms and labs, run them, analyze their output, modify them and rerun them easily. This makes mathematics more relevant and motivating for engineering students and provides valuable insight into the underlying dynamics, which helps the learner of mathematics to gain a better feel for what is going on than has hitherto been possible. The purpose of this article is to demonstrate how Maple can be used to investigate and visualize with animations a selection of examples

from our engineering mathematics curriculum [1, 2, 3], Taylor polynomials for $f(x, y)$, the motion of a simple pendulum, a system of two or three spring-coupled masses, reflection and transmission of a wave pulse at a boundary, graphical interpretation of some convolutions and graphical visualizing of the solutions of a discrete dynamical system described by a difference equation.

Taylor Polynomial in two variables

If we know the value of $f(x, y)$ and its partial derivatives at a point $x = a, y = b$, then the Taylor polynomial in two variables allow us to estimate $f(x, y)$ at points near to (a, b) . We define the Taylor polynomial of degree n generated by f at $x = a, y = b$ as a function in the same way as the polynomial in one variabel by:

> **restart:**

> **readlib(mtaylor):**

> **P:=(f,a,b,n,u,v)->subs(x=u,y=v,mtaylor(f,[x = a,y=b],n+1)):**

The Taylor polynomial of degree 2 at (a, b) is:

> **'P'(f(x,y),a,b,2,x,y)'=P(f(x,y),a,b,2,x,y);**

$$P(f(x, y), a, b, 2, x, y) = f(a, b) + D_1(f)(a, b) (x - a) + D_2(f)(a, b) (y - b) + \frac{1}{2} D_{1,1}(f)(a, b) (x - a)^2 + (x - a) D_{1,2}(f)(a, b) (y - b) + \frac{1}{2} D_{2,2}(f)(a, b) (y - b)^2$$

With $f(x, y) = 2 + \cos(x) + \sin(y)$, $(a,b) = (0, 2)$ and $n = 2$ we get the following approximating to $f(x, y)$ at the point $(3, 3)$.

> **f:=(x,y)->2+cos(x)+sin(y):**

> **'P'(f(x,y),0,2,2,x,y)=P(f(x,y),0,2,2,x,y);**

$P(2 + \cos(x) + \sin(y), 0, 2, 2, x, y) =$

$$3 + \sin(2) + \cos(2) (y - 2) - \frac{1}{2} x^2 - \frac{1}{2} \sin(2) (y - 2)^2$$

> **'P'(f(3,3),0,2,2,1,-1)=evalf(P(f(x,y),0,2,2,1,-1));**

$$P(2 + \cos(3) + \sin(3), 0, 2, 2, 1, -1) = .565899516$$

> **f(3,3)=evalf(f(1,-1));**

$$2 + \cos(3) + \sin(3) = 1.698831321$$

> **'P'(f(3,3),0,2,12,1,-1)=evalf(P(f(x,y),0,2,12,1,-1));**

$$P(2 + \cos(3) + \sin(3), 0, 2, 12, 1, -1) = 1.698777264$$

With $n = 12$ the values of the Taylor polynomial and the funtion are in close agreement, which is visualized in

Figure 1.

> **f:=(x,y)->2+cos(x)+sin(y):**

- **Taylor3D(f, a, b, n)** animates a Taylor polynomial of dgree **n** generetated by $f(x, y)$ at $x = a, y = b$.

> **Taylor3D(f,0,2,12,x=-2..2,y=-4..4,z=0..4,axes=frame,orientation=[-19,86]);**

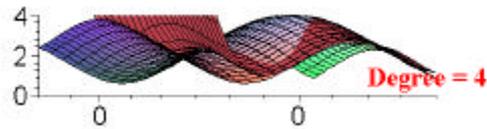


Figure 1 Animation of Taylor polynomials generated by $f(x, y) = 2 + \cos(x) + \sin(y)$ at $(0, 2)$

Simple Pendulum

The objective in this section is to visualize the motion of a simple pendulum which consist of a small mass m suspended by a light inextensible cord of length L from a fixed support. If we apply the law of the conservation of mechanical energy to analyze the motion of the mass m , we obtain the following initial value problem described by the differential equation

$$\frac{\partial^2}{\partial t^2} \theta(t) + \frac{g}{L} \sin(\theta) = 0$$

subject to the initial conditions $\theta(0) = \theta_0$ and $\left. \left(\frac{\partial}{\partial t} \theta(t) \right) \right|_{t=0} = v_0$. g is the acceleration of gravity and the angular displacement $\theta(t)$ is measured from the vertical. If we account for the frictional resistance of the surrounding medium which is proportional

to the instantaneous velocity, $c \frac{\partial}{\partial t} \theta(t)$, the result is the differential equation:

$$\frac{\partial^2}{\partial t^2} \theta(t) + c \frac{\partial}{\partial t} \theta(t) + \frac{g}{L} \sin(\theta) = 0$$

This equation has no closed-form solution for $\theta(t)$, so we find a numerical solution using the Runge-Kutta-Fehlberg fourth-fifth method. We define the general nonlinear pendulum equation by:

> **deq:=(c,L)->diff(theta(t),t\$2)+c*diff(theta(t),t)+9.8/L*sin(theta(t))=0:**

With the damping coefficient $c = .2 s^{(-1)}$, $L = 5 m$ and the initial conditions $\theta(0) = \frac{\pi}{3}$

, $D(\theta)(0) = 0$ we get:

> **sol:=dsolve({deq(0.2,5),theta(0)=Pi/3,D(theta)(0)=0},theta(t),numeric,startinit=true):**

The graphs of $\theta(t)$ and $\frac{\partial}{\partial t} \theta(t)$ is shown in Figure 2.

> **with(plots):**

> **plt1:=odeplot(sol,[t,theta(t)],0..10):**

plt2:=odeplot(sol,[t,diff(theta(t),t)],0..10,color=blue):

> **display(plt1,plt2);**

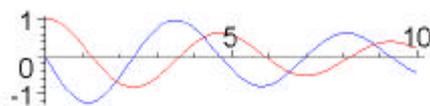


Figure 2 Angular and angular velocity position for the simple pendulum with damping

The pseudoperiod T_1 of time for the first complete damped oscillation is about four times the amount of time required for θ to decrease from $\theta = \theta_0$ to $\theta = 0$.

> **alpha:=u->subs(sol(u),theta(t)):**

> **T[1]=4*fsolve('alpha(u)'=0,u=0..2);**

$$T_1 = 5.057414828$$

Figure 3 animates damped motion of the pendulum and the simultaneous position of the mass on the angular position curve.

- **Pendulum(L, q₀, c)** create an animation of the motion of a simple pendulum. The angular position θ is controlled by the numerical solution of the differential equation describing the pendulum's motion. The initial angular velocity is assumed zero. **L**: length of the pendulum; **q₀**: initial angular position in degrees; **c**: damping coefficient.

> **Pendulum(5,60,0.2,scaling=constrained,axes=normal,tickmarks=[4,4]);**

$$\left(\frac{\partial^2}{\partial t^2} \theta(t)\right) + .2 \left(\frac{\partial}{\partial t} \theta(t)\right) + 1.960000000 \sin(\theta(t)) = 0$$

Pseudoperiod
 $T = 5.057414828$

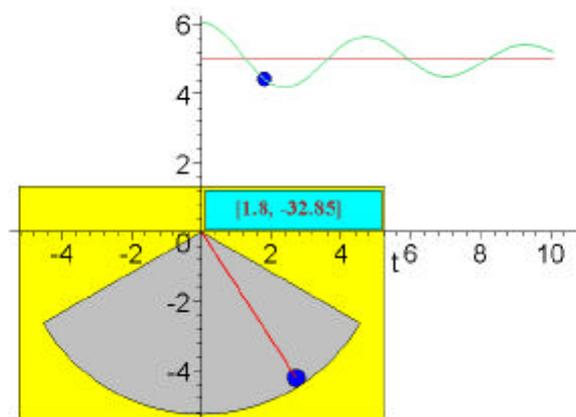


Figure 3 Damped motion of a simple pendulum

Spring-Coupled Masses

The purpose of this section is to describe and animate the motion of two or three masses connected to attached springs. Figure 4 shows three masses, m_1 , m_2 and m_3 connected to each other and to two walls by four springs with spring constants k_1 , k_2 , k_3 and k_4 .

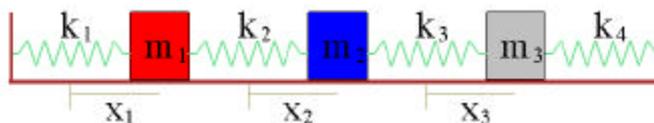


Figure 4 Three spring-coupled masses

Since friction is to be neglected, the only forces acting on the masses are those due to the extension and compression of the attached springs. We take the rightward

displacements x_1, x_2 and x_3 of the respective masses from their equilibrium positions as coordinates. The first spring is then stretched the distance x_1 , the second spring is stretched the distance $x_2 - x_1$, the third spring is stretched the distance $x_3 - x_2$ and the fourth spring is stretched the distance x_3 . If we assume that each spring obeys Hooke's law, Newton's law gives us the following set of differential equations:

$$\begin{aligned} m_1 \frac{\partial^2}{\partial t^2} x_1(t) &= -k_1 x_1(t) + k_2 (x_2(t) - x_1(t)) \\ m_2 \frac{\partial^2}{\partial t^2} x_2(t) &= -k_2 (x_2(t) - x_1(t)) + k_3 (x_3(t) - x_2(t)) \\ m_3 \frac{\partial^2}{\partial t^2} x_3(t) &= -k_3 (x_3(t) - x_2(t)) - k_4 x_3(t) \end{aligned}$$

In matrix form, this system can be written:

$$\mathbf{M} \mathbf{x}''(t) = \mathbf{K} \mathbf{x}(t)$$

where the \mathbf{M} is the mass matrix and \mathbf{K} the stiffness matrix. Then it follows

$$\mathbf{x}''(t) = \mathbf{A} \mathbf{x}(t)$$

where $\mathbf{A} = \mathbf{M}^{-1} \mathbf{K}$.

If the 3×3 matrix \mathbf{A} has distinct negative eigenvalues $-\omega_1^2, -\omega_2^2, -\omega_3^2$ with associated real eigenvectors $\mathbf{V}_1, \mathbf{V}_2$ and \mathbf{V}_3 , then the solution of $\mathbf{x}''(t) = \mathbf{A} \mathbf{x}(t)$ is given by

$$\mathbf{x}(t) = \sum_{i=1}^3 (a_i \cos(\omega_i t) + b_i \sin(\omega_i t)) \mathbf{V}_i$$

- **SpringMassCouplet(L, init)** solves the second order system described above, displays the matrix

$\mathbf{A} = \mathbf{M}^{-1} \mathbf{K}$, the eigenvalues and the corresponding circular frequencies. The motion of the spring-couplet masses can be animated in four different cases:

- 1) Three masses and four springs: $\mathbf{L} = [[m_1, k_1], [m_2, k_2], [m_3, k_3], k_4]$
- 2) Three masses and three springs, no spring connected to the right-hand wall: $\mathbf{L} = [[m_1, k_1], [m_2, k_2], [m_3, k_3]]$
- 3) Two masses and three springs: $\mathbf{L} = [[m_1, k_1], [m_2, k_2], k_3]$
- 4) Two masses and two springs, no spring connected to the right-hand wall: $\mathbf{L} = [[m_1, k_1], [m_2, k_2]]$

The initial conditions is defined by $\mathbf{init} = [x_1(0), x_2(0), x_3(0), D(x_1)(0), D(x_2)(0), D(x_3)(0)]$

Three masses and four springs

> $\mathbf{L} := [[1,1],[1,2],[1,2],3]; \mathbf{init} := [-1.5,1,0,0,0,0];$

> **SpringMassCouplet(L,init,10,scaling=unconstrained,axes=frame);**

$$\left(\frac{\partial^2}{\partial t^2} x_1(t) \right) + 3 x_1(t) - 2 x_2(t) = 0,$$

$$\left(\frac{\partial^2}{\partial t^2} x_2(t) \right) + 4 x_2(t) - 2 x_1(t) - 2 x_3(t) = 0,$$

$$\left(\frac{\partial^2}{\partial t^2} x_3(t)\right) + 5 x_3(t) - 2 x_2(t) = 0$$

coefficient matrix

$$\begin{bmatrix} \frac{1}{m_1} & 0 & 0 \\ 0 & \frac{1}{m_2} & 0 \\ 0 & 0 & \frac{1}{m_3} \end{bmatrix} \begin{bmatrix} -k_1 - k_2 & k_2 & 0 \\ k_2 & -k_2 - k_3 & k_3 \\ 0 & k_3 & -k_3 - k_4 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 0 \\ 2 & -4 & 2 \\ 0 & 2 & -5 \end{bmatrix}$$

eigenvalues

$$[-\omega_1^2, -\omega_2^2, -\omega_3^2] = [-4., -7., -1.]$$

natural frequencies

$$[\omega_1, \omega_2, \omega_3] = [2.000000000, 2.645751311, 1.000000000]$$

$$\begin{aligned} x_1(t) = & -0.8888888889 \cos(2. t) - 0.3888888889 \cos(2.645751311 t) \\ & - 0.2222222222 \cos(t), x_2(t) = 0.4444444444 \cos(2. t) \\ & + 0.7777777778 \cos(2.645751311 t) - 0.2222222222 \cos(t), x_3(t) = \\ & 0.8888888889 \cos(2. t) - 0.7777777778 \cos(2.645751311 t) \\ & - 0.1111111111 \cos(t) \end{aligned}$$

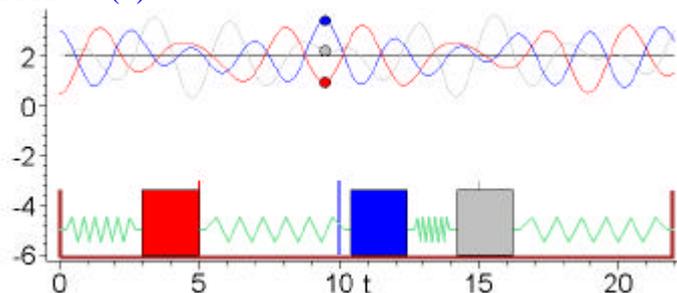


Figure 5 Three spring-couplet masses.

The vertical lines marks the equilibrium positions of the masses.

$$m_1 = m_2 = m_3 = 1 \text{ kg}, k_1 = 1 \frac{N}{m}, k_2 = k_3 = 2 \frac{N}{m} \text{ and } k_4 = 3 \frac{N}{m}.$$

Figure 5 shows that the natural frequencies of the system is $[\omega_1, \omega_2, \omega_3] = [2.000000000, 2.645751311, 1.000000000]$. In the first natural mode the two masses m_1 and m_3 move in opposite directions with equal amplitudes. The mass m_2 move in the same direction but with the amplitude of motion half that of m_3 . In the second mode m_1 and m_3 move in the same directions, opposite to m_2 , with the amplitude m_3 twice that of m_1 and equal to that of m_2 . In the last mode with frequency ω_3 all three masses move in the same direction. m_1 and m_2 have equal amplitudes twice that of m_3 .

Reflection/Transmission of Wave Pulses

Here we want to visualize the reflection and the transmission of a transverse wave

pulse at the boundary of two ropes of different linear densities.

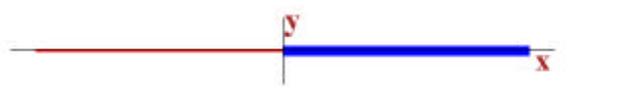


Figure 6 Two ropes of different linear densities μ_1 and μ_2

Suppose that the two ropes in Figure 6, having masses per unit length of μ_1 and μ_2 lie along the x -axes in their equilibrium position and are joined at the origin, $x = 0$. It is easy to show that the equation $y = f(x - c_1 t)$ describes a pulse with function f traveling to the right with speed c_1 . $y = f(x + c_2 t)$ describes a pulse traveling to the left with speed c_2 . Let us suppose that some point of a stretched rope is forced to oscillate transversely with simple harmonic motion such that a continuous succession of pulses, or a continuous wave train, travels along the rope. Any transmitted or reflected wave must have the same frequency ν as the incident wave at the boundary of the two ropes. For the incident, reflected and transmitted wave the displacement at any time is given by respectively

$$y_i = A_i \sin 2\pi\nu \left(t - \frac{x}{c_1} \right), y_r = A_r \sin 2\pi\nu \left(t + \frac{x}{c_1} \right), y_t = A_t \sin 2\pi\nu \left(t - \frac{x}{c_2} \right)$$

where A_i , A_r and A_t are the amplitudes of the incident, reflected and transmitted waves respectively. At the boundary, $x = 0$, the vertical displacement of the two ropes must be the same at every instant of time, or $y_i + y_r = y_t$. This gives that $A_i + A_r = A_t$. At the boundary, $x = 0$, the vertical forces on the ropes must be the same. If the tension F in the two ropes is the same it follows that

$$F \left(\left(\frac{\partial}{\partial x} y_i \right) + \left(\frac{\partial}{\partial x} y_r \right) \right)_{x=0} = F \left(\frac{\partial}{\partial x} y_t \right)_{x=0} \quad \text{and subsequently} \quad \frac{A_i}{c_1} - \frac{A_r}{c_1} = \frac{A_t}{c_2}$$

$$\frac{A_i - A_r}{A_i + A_r} = \frac{c_1}{c_2}$$

If we define the coefficient of reflection R as the ratio of the amplitude of the

reflected wave to the incident wave, then $R = \frac{A_r}{A_i} = \frac{c_2 - c_1}{c_2 + c_1}$. The velocities in the

two ropes are $c_1 = \sqrt{\frac{F}{\mu_1}}$ and $c_2 = \sqrt{\frac{F}{\mu_2}}$. From this it follows that

$$R = \frac{\sqrt{\mu_1} - \sqrt{\mu_2}}{\sqrt{\mu_1} + \sqrt{\mu_2}}$$

- **WavePulse(m_1, m_2)** animates reflection and transmission of a transverse pulse at the boundary of two ropes of different linear density m_1 and m_2

Let us illustrate what happens to a wave pulse if :

- a) The mass per unit length μ_1 of rope 1 is smaller than μ_2 of rope 2.

> **plt1:=WavePulse(1,4,axes=none,scaling=unconstrained,title='a')**:

Figure 7 a) shows that the amplitude of the reflected pulse A_r is opposite to that of

the incident wave because of the negative value of the coefficient of reflection, R , according to the equation above.

b) If we let $\frac{\mu_1}{\mu_2} \rightarrow 0$ then $R \rightarrow -1$ and $A_r \sim -A_i$ and $A_t \sim 0$ as illustrated in Figure 7 b)

where $\mu_1 \ll \mu_2$.

> **plt2:=WavePulse(1,10^8,axes=None,scaling=unconstrained,title='b'):**

c) The mass per unit length μ_1 of rope 1 is greater than μ_2 of rope 2.

> **plt3:=WavePulse(4,1,axes=None,scaling=unconstrained,title='c'):**
display(array([plt1,plt2,plt3]));

When $\mu_2 < \mu_1$ R is positive. There will be no change in phase at reflection, A_r has the same sign as A_i , as

Figure 7 c) shows.

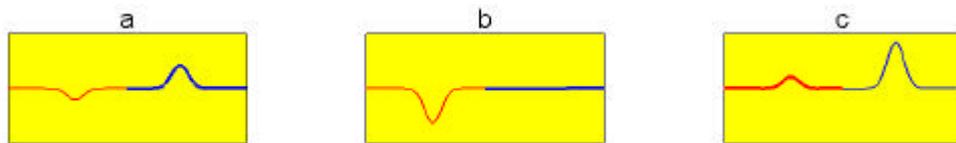


Figure 7

The red rope has the density μ_1 , the blue rope the density μ_2 , a) Wave pulse advancing along two ropes of different linear densities, $\mu_1 = 1$, $\mu_2 = 4$, b) Wave pulse reflected at the boundary of two ropes, linear densities, $\mu_1 = 1$, $\mu_2 = 10^8$, c)

Wave pulse advancing along two ropes, $\mu_1 = 4$, $\mu_2 = 1$

Convolution

The convolution of $f(t)$ and $g(t)$, denoted by $f * g(t)$, is given by

$$f * g(t) = \int_0^t f(\tau) g(t - \tau) d\tau$$

Convolution is a useful concept and can be found in various places in applied mathematics since it plays an important role in for instance heat conduction, wave motion and time series analysis. Here we intend to give an interpretation of $f * g$ by animation of the convolution product of a rectangular window $f(t) = u(t) - u(t - 1)$ with an exponential function $g(t) = e^{(-t)}$.

> **restart: alias(u=Heaviside):**

Let us define the convolution product $h = f * g$ by the integral

> **h:=t->Int(f(tau)*g(t-tau),tau=0..t):**

> **'h(t)'=h(t);**

$$h(t) = \int_0^t f(\tau) g(t - \tau) d\tau$$

The convolution of $f(t) = u(t) - u(t - 1)$ and $g(t) = e^{(-t)}$, where $u(t)$ is the Heaviside unit step function, is given by $h(t)$.

> **f:=t->u(t)-u(t-1):g:=t->exp(-t):**

> **'h(t)=h(t);**

$$h(t) = \int_0^t (u(\tau) - u(\tau - 1)) e^{-(t+\tau)} d\tau$$

The graph of this convolution product is shown in Figure 8

> **plot(value(h(t)),t=0..8,labels=['t', 'h(t)'],color=blue);**

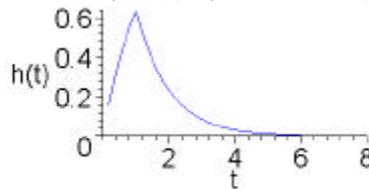


Figure 8 The convolution product $h(t) = \int_0^t f(\tau) g(t - \tau) d\tau$,
 $f(t) = u(t) - u(t - 1)$, $g(t) = e^{(-t)}$

• **Convolution(f, g)** animates the convolution product $f * g$

Figure 9 shows the animation of the convolution product of the rectangular window $f(t)$ and the exponential function $g(t)$. The counter displays the value of the overlapping area between the green exponential area and the red windows as the red window move from left to right towards the stationary green area. The green disk

moves on the blue convolution curve $h(t) = \int_0^t f(\tau) g(t - \tau) d\tau$ and indicate

graphically the value of $h(t)$.

> **Convolution(f,g);**

$$\int_0^t f(\tau) g(t - \tau) d\tau = u(t) - e^{(-t)} u(t) - u(t - 1) + u(t - 1) e^{(1-t)}$$

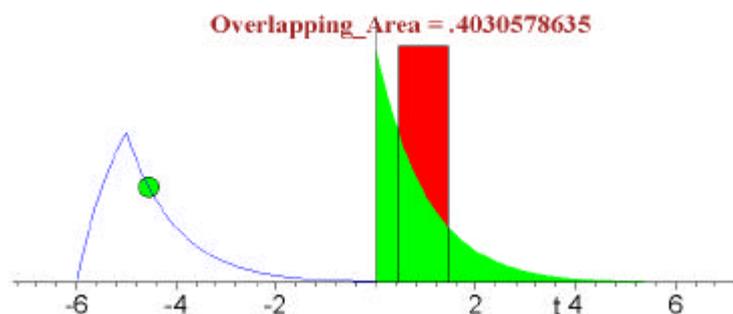


Figure 9 Animation of the convolution product $f * g$.

$$f(t) = u(t) - u(t - 1), \quad g(t) = e^{(-t)}$$

The convolution product reach a maximum equal to $\int_0^1 e^{(-t)} dt = .6321205588$ when the right hand side of the red rectangle is at $t = 1$, and the green disk is at the top of the convolution curve $h(t)$.

Discret Dynamical System

A discrete dynamical system is a sequence of numbers that are defined recursively. Eigenvalues and eigenvectors provide the key to understanding the evolution of a dynamical system described by a difference equation $\mathbf{X}(k+1) = A \mathbf{X}(k)$ where A is a square matrix. We can view this equation as a description of what happens to an initial point $\mathbf{X}(0)$ in the plane as it is transformed repeatedly by the mapping $\mathbf{X} \rightarrow A \mathbf{X}$. The long-term behaviour of the dynamical system is equivalent of the steady-state response in an engineering control system. An instructive way to see how multiplication by the matrix A affects points is to plot an arbitrary point $\mathbf{X}(0)$ and then to plot successive images of this point under repeated multiplications by A .

- **PointIteration**($A, \mathbf{X0}, N$) animates trajectories of the system $\mathbf{X}(k+1) = A \mathbf{X}(k)$. $\mathbf{X0}$ is the initial point $\mathbf{X}(0)$.

Rotation

Let $A = \begin{bmatrix} .5 & -.6 \\ .75 & 1.1 \end{bmatrix}$.

> **A:=matrix(2,2,[0.8,0.5,-0.1,1.0]); LX0:=[[0,2.5],[4,0],[0,-2.5]]:**

Figure 12 animates three points under the action of the matrix A with complex eigenvalues

> **PointIteration(A,LX0,45,line,view=[-5..5,-3..3]);**

$$A = \begin{bmatrix} .8 & .5 \\ -.1 & 1.0 \end{bmatrix}$$

Eigenvalues

$$[\lambda_1, \lambda_2] = [.9000000000 + .2000000000 I, .9000000000 - .2000000000 I]$$

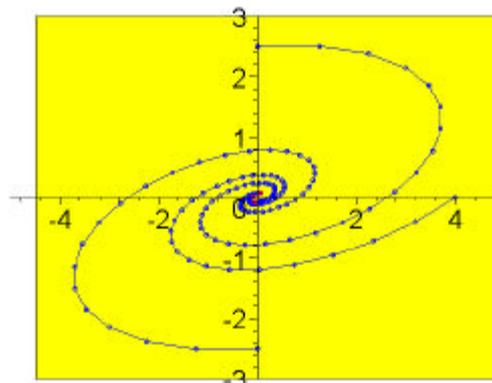


Figure 12 Iteration of the points $[0,2.5],[4,0],[0,-2.5]$ under the action of

$$A = \begin{bmatrix} .8 & .5 \\ -.1 & 1.0 \end{bmatrix}$$

References

- [1] C. H. Edwards and D. E. Penney. *Calculus with Analytic Geometry*, Prentice Hall International, Inc., 1998.
- [2] C. H. Edwards and D. E. Penney. *Differential Equations and Boundary Value Problems. Computing and Modeling.*, Prentice Hall International, Inc., 1996.
- [3] D. C. Lay. *Linear Algebra and its Applications, Second Edition*, Addison Wesley, 1997.