Graphics Calculators: Some Implications for Course Content and Examination

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Abstract

In this paper we consider some changes that the use of graphics calculators imposes on the teaching and examination of calculus and linear algebra. We examine the type of material that could be omitted from the present syllabus, as well as some topics that would need to be added. In the latter category we include some description of the algorithms calculators use to construct graphs, perform symbolic differentiation and integration and solve systems of linear equations. Particular reference will be made to the HP 38G and HP 48G graphics calculators.

1 Introduction

In recent years the availability and affordability of graphics calculators have increased dramatically. This calculator availability will have an enormous effect not only on the way mathematics is taught but also on what mathematics is taught and how this is assessed. In this paper we shall consider some implications for the mathematics curriculum, and its assessment, at the upper secondary/lower tertiary levels and we shall illustrate our comments with examples from the areas of calculus and linear algebra.

It it tempting for students to regard a graphics calculator as a 'black box' and not to be concerned with how it works. However, this approach can lead to serious errors so it is important that students be taught how to use the graphics calculator intelligently, rather than simply be told which buttons to press to perform a particular task. This requires some knowledge of the various algorithms used and an understanding of some of the limitations involved. We shall illustrate this view in the following sections, with particular reference to the HP 38G and HP 48G graphics calculators.

2 Calculus

In calculus there are two major areas where graphics calculators are useful. The first of these is the graphing of functions, including the use of the graphs to calculate roots and find extrema. Often the functions to be graphed are ones with which students are not familiar. This is particularly the case when they are engaged in mathematical modelling. It is therefore essential that students have some understanding of the graphing approach taken by the calculator and a realization that the picture obtained may not be exact.

The second is the evaluation of derivatives and integrals, both symbolic and numerical. Numerical applications of calculus, in particular, can be performed on a graphics calculator with only a minimal understanding of the operations involved. Certainly, there is room for mistakes or confusion if an integral has improper limits, or if the student is unaware that a numerical answer is typically just an approximation. In the case of the HP 48G these pitfalls are discussed in detail in the User's Guide.

Symbolic differentiation and integration are altogether different. As we shall see, they require far greater care in both teaching and application.

2.1 Graphing functions

For the inexperienced student the ability to generate a graph via the graphics calculator is a vital asset. However, it is important for students to realise that the graphics calculator generates a graph by plotting points and joining these by straight line segments. The HP 38G gives the user the options **Faster** and **More Detail**. The former simply plots fewer points than the latter. The HP 48G gives the user the choice of increment size. One can obtain an initial picture by choosing (virtually at random) a domain and range for the calculator to work with and then making use of the various rescaling options to get a nicer picture. However real confidence in this procedure relies on the student having some idea of the general shape of the graph under consideration.

In addition, both the HP 38G and the HP 48G have an **Autoscale** option which, although it is often quite convenient, particularly on the HP 38G for statistical plots, is not the panacea that one might expect. This option carries out an autoscaling by an automatic scaling of the vertical axis but only on the default domain currently in the calculator memory. Hence, rather than starting from scratch, the picture obtained depends on what domain was considered previously. When using the PLOT option on the HP 48G the default domain is given on the PLOT screen and the use of Autoscale produces the transient message *Autoscale vertical plot range.* Neither of these advantages is provided by the HP 38G.

Example 1 *Limits using the graphs of* $\sin(1/x)$ *and* $x\sin(1/x)$*.*

For $\sin(1/x)$ with initial domain$[-1,1]$ one quickly notes the oscillations in the neighbourhood of $x=0$. Using rescaling or a **Zoom** option with domains $[-0.1,0.1]$ and subsequently $[-0.01,0.01]$ one notices that these oscillations increase in number and one is led to the conclusion that $\lim_{x\to 0}\sin(1/x)$ does not exist and one can use this information to formulate a formal proof. A change of domain to $[-10,10]$ allows one to conclude that $\lim_{x\to\pm\infty}\sin(1/x)=0$.

For $x\sin(1/x)$ a glance at the graph on the domain $[-1,1]$ indicates that $\lim_{x\to 0} x\sin(1/x)=0$. The more aware student will realise that the graph of $x\sin(1/x)$ is actually bounded above by the graph of $|x|$ and below by the graph of $-|x|$. This can be then used to give a formal proof, using the Squeeze Law, of the value of the above limit. A change of domain to $[-10,10]$ allows one to conclude that $\lim_{x\to\infty} x\sin(1/x)=1$.

One should, however, note that, in each of the above cases, the graphs obtained are not strictly accurate. For $\sin(1/x)$ the oscillations do not all reach -1 and 1 and for $x\sin(1/x)$ they do not all reach $-|x|$ and $|x|$. This limitation does not affect us here but such details might well be of crucial importance in a different context with these or other functions.

Classroom access to a graphics calculator enables students to consider more readily real-world situations involving functions that may be outside their current experience. One example is the consideration of a simple model of a rainbow [?] which results in the function $y = 4\arcsin((\sin x)/\mu) - 2x$, where y is the angle between the direction of the incoming ray and the direction of the outgoing ray (known as the scattering angle), x is the angle of incidence of the ray and μ is its index of refraction. For a given value of μ the outgoing ray will be observed by the eye when y is a maximum with respect to x. The resulting function can readily be graphed on a graphics calculator and the maximum obtained. In practice, at the precalculus level, one would like x and y to be given in degrees rather than radians. On the HP 38G and the HP 48G this can be easily achieved by ensuring that, within the appropriate plotting sequence, the angle mode is set to degrees.

2.2 Symbolic differentiation

Symbolic differentiation presents few problems on a graphics calculator. A graphics calculator can differentiate symbolically any function a high-school student can, and is infinitely preferable to pen and paper when it comes to differentiating elaborate quotients or compositions of functions.

One limitation of symbolic differentiation routines is that a derivative can take completely different forms if the original function is entered in only slightly different ways. For example, if $f(x) = x^{-2}$ is entered as "X^–2", both the HP 38G and HP 48G use the power rule to return f' as "–(2*X^–3)"; whereas if f is entered as "1/X^2", they use the quotient rule to give

"–(2*X/X^2^2)". This would not be a problem if the calculators could be used to demonstrate the equivalence of the two expressions. Unfortunately neither calculator has the capability to reduce "–(2*X/X^2^2)" to the more natural form "–(2*X^–3)", and students require a certain level of algebraic awareness to recognise the equivalence.[1]

Care should also be exercised when symbolically differentiating exponential functions. The HP 48G returns the derivative of $g(x) = e^x$ as "EXP(X)" as expected. However, the answers generated by the HP 38G require more attention. On the Home display g' is returned as "2.71828182846^X" (the actual value of the numerical prefix depends on the format used), whereas in the Function Symbolic View it is given as "LN(e)*e^X". Both calculators return the derivative of 2^x as ".69314718056*2^X" rather than the more conventional $\ln 2 \cdot 2^x$. The student therefore needs to be familiar with some of the standard mathematical constants.

2.3 Symbolic integration

Symbolic integration on a graphics calculator needs to be approached with more caution. In the first place, the library of functions that a graphics calculator can integrate symbolically is rather small. In the HP 38G and HP 48G it consists almost exclusively of standard functions with arguments that are no worse than linear in the integration variable, plus sums and differences of such functions. Neither calculator can make simple substitutions or integrate by parts. As a consequence, the calculators should only be used as

[1] The HP 48G does have a "Collect" command (found by pressing [⇐] [SYMBOLIC] {{COLCT}}) which groups together powers of like terms and reduces them to a single exponent, but it is not effective on expressions such as "–(2*X/X^2^2)".

an adjunct to a student's integration skills, not as a replacement for them. They are perhaps most effective if treated in the same way as an elementary table of integrals.

In the second place, the lack of a preliminary algebraic simplification routine in either calculator means that they can stumble over an integral like $\int (x/x)\, dx$. This can thwart more elaborate but straightforward integrals such as $\int_1^x (1+u)/u\, du$, which both the HP 38G and HP 48G return as "LN(X)+∫(1,X,U/U,U)". Fortunately, the second term can be evaluated explicitly on the HP 48G by using the "Collect" command.

Another limitation is that the ability of a graphics calculator to integrate a function can depend critically on how the function is written. The following table gives a list of sample functions the HP 38G can integrate symbolically, together with equivalent or similar forms which it cannot integrate.

The HP 38G can integrate...	But it cannot integrate...
$(ax+b)^{-k}$	$1/(ax+b)^k$ if $k \neq 1$ or 2
$1/[1 \pm (ax)^2]$	$[1 \pm (ax)^2]^{-1}$ or $1/(1 \pm a^2x^2)$
	$1/[b^2 \pm (ax)^2]$ if $b^2 \neq 1$
$1/\surd[1 \pm (ax)^2]$	$1/[1 \pm (ax)^2]^{1/2}$ or $[1 \pm (ax)^2]^{-1/2}$
	$1/\surd[(ax)^2 - 1]$ in any form
$\tan x$ and $\tanh x$	$\sin x/\cos x$ or $\sinh x/\cosh x$
$1/\tan x$	$(\tan x)^{-1}$, $\cot x$ or $\cos x/\sin x$
$1/(\sin x \cos x)$	$(\sin x \cos x)^{-1}$ or $\sin x \cos x$
$\tan^2 x$	$\sin^2 x$, $\cos^2 x$ or $\tanh^2 x$
$\exp(-x)$	$1/\exp x$ or $(\exp x)^{-1}$

The list is by no means exhaustive, but it seems clear that there is no simple general rule governing the syntax of a successful integrand on this calculator. Although the HP 48G shares the same symbolic integration software as the HP 38G, the added feature of the "Collect" command makes the HP 48G more versatile, enabling it to integrate the expressions $1/(ax+b)^k$, $[1 \pm (ax)^2]^{-1}$, $(\tan x)^{-1}$ and $(\sin x \cos x)^{-1}$.

When integrating exponentials on a graphics calculator the same care needs to be exercised over mathematical constants as was recommended earlier in connection with symbolic differentiation. For example, the HP 48G returns the primitive of e^x as "EXP(X)" as expected, but the HP 38G gives "e^X/LN(e)". Both the HP 38G and HP 48G return the primitive of 2^x as "2^X/.69314718056" rather than $2^x/\ln 2$.

In summary, there are very few general rules that can be offered to a student who wants to use the HP 38G or HP 48G to perform symbolic integration. When attempting to integrate rational functions it is essential that the quotient be reducible (by hand) to a polynomial plus a sum of terms of the form $c/(ax+b)$ or $c/[1 \pm (ax)^2]$. Any obvious substitution of variables will also need to be done by hand. Otherwise, if an attempted symbolic integration is unsuccessful the best course of action is to try as many simple equivalent forms of the integrand as possible, and (if the HP 48G is being used) to use the "Collect" and/or EVAL commands at every opportunity.

3 Linear Algebra

When working with matrices a lot of time is taken up by routine calculations which may well be relegated to a machine once the principles involved have been understood by the students. Calculators such as the HP 38G and the HP 48G are well suited for this purpose as they provide a large number of algorithms for work with matrices in addition to matrix arithmetic.

3.1 Solving systems of linear equations

Many standard scientific calculators will solve systems of linear equations $A\mathbf{x} = \mathbf{b}$ in two or three unknowns whose coefficient matrix A is nonsingular. This is usually handled by calculating the inverse A^{-1} of the matrix and then premultiplying both sides of the equation $A\mathbf{x} = \mathbf{b}$ by A^{-1}. While this is the technique that most closely mimics that for the solution of a linear equation in one variable, it excludes from consideration all those systems of linear equations whose coefficient matrix is singular or non-square. For an in-depth discussion of the solution theory for systems of linear equations a method such as Gauss-Jordan elimination is more appropriate. The HP 48G and the HP 38G both provide a program called RREF which calculates the reduced row echelon form (rref) of a matrix. Its use enables one to expose students to a wider variety of examples than previously, making the discussion of the solution theory more accessible.

Example 2 *Consider the systems of linear equations* $A\mathbf{x}_1 = \mathbf{b}_1$ *and* $A\mathbf{x}_2 = \mathbf{b}_2$ *where*

$$A = \begin{bmatrix} -8 & 6 & 0 \\ 4 & -12 & 3 \\ 4 & 6 & -3 \end{bmatrix}, \mathbf{b}_1 = \begin{bmatrix} 8 \\ -2 \\ -6 \end{bmatrix} \text{ and } \mathbf{b}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Here a calculation of the rref of the augmented matrix $[A \mid \mathbf{b}_1 \mid \mathbf{b}_2]$ shows that the first system of equations has infinitely many solutions, while the second has no solution.

While the process of determining the rref of a matrix has been made less arduous, the trade-off is having to address the effects of rounding. When the coefficient matrix has irrational entries or rational entries whose decimal expansion is non-terminating, rounding errors may easily lead to incorrect answers.

Example 3 *Solve the homogeneous system of linear equations* $B\mathbf{x} = \mathbf{0}$ *where*

$$B = \begin{bmatrix} -\frac{2}{3} & \frac{1}{2} & 0 \\ \frac{1}{3} & -1 & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{2} & -\frac{1}{4} \end{bmatrix}.$$

We note that $12B = A$. Thus the reduced row echelon forms of A and B are equal. However both the HP 38G and HP 48G return $[I_3 \mid \mathbf{0}]$ as the rref of $[B \mid \mathbf{0}]$ which would tell us that B is invertible and that $\mathbf{x} = \mathbf{0}$ is the only solution to this system of equations. The cause for this incorrect result is rounding; since the calculator works with a finite decimal expansion only, the difference $2/3 - 2 \times (1/3)$ is equated to .000000000001. Multiplying the augmented matrix by a suitable constant, in this case 6, results in finding the correct rref. This salvage approach will not work when the matrix contains irrational entries.

The HP 48G also provides an environment called SOLVE which can be used for solving systems of linear equations. This application is very powerful. The inputs are the coefficient matrix A and the constant vector $\mathbf{b}$. The output is a solution vector $\mathbf{x}$, which will represent the unique solution to the system of equations when the coefficient matrix is invertible, the solution vector of minimum Euclidean norm when the system is consistent but has infinitely many solutions, and the least squares solution when the system of linear equations is inconsistent. This feature unfortunately renders SOLVE useless in the case of a homogeneous system of equations as the application will always return the zero-vector as the solution vector. If SOLVE is used to determine the solution of $A\mathbf{x}_1 = \mathbf{b}_1$ and $A\mathbf{x}_2 = \mathbf{b}_2$ with A, $\mathbf{b}_1$ and $\mathbf{b}_2$ from Example 2, we obtain $\mathbf{x}_1 = (-1.089, -.118, .312)^T$ and $\mathbf{x}_2 = (7.495E-2, 4.438E-2, -3.353E-2)^T$ without any indication that $\mathbf{x}_1$ is not unique or that $\mathbf{x}_2$ is a least-squares solution.

3.2 Finding eigenvalues and eigenvectors

Both the HP 38G and the HP 48G provide programs for calculating eigenvalues and eigenvectors of square matrices. The routine used to calculate the eigenvalues and eigenvectors is a power method. Thus the eigenvectors will be scaled in so that the entry of greatest magnitude will be 1 or -1, which often leads to unwieldy fractions as the components of the eigenvectors. This is more problematic with the HP 48G than the HP 38G which has a fractions mode that will return the eigenvectors in fraction format to an accuracy determined by the user.

The routines tend to work well when the eigenvalues of the matrix are distinct. However when determining eigenvectors for eigenvalues of multiplicity greater than 1, the eigenvectors calculated by the algorithms can at times be counter-intuitive. For example the matrix

$$\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

has eigenvalues 1 and 2, the latter having algebraic and geometric multiplicity 2. The eigenvectors corresponding to 2 are found to be $(0,1,0)^T$ and $(-1,-.22360679775,1)^T$. These are clearly linearly independent, however the entry in the second component of the second eigenvector would have been set to 0 in a hand-calculation. If the given matrix has an eigenvalue whose algebraic and geometric multiplicities differ, the HP 48G and the HP 38G will calculate a basis for the eigenspace corresponding to that eigenvalue and then list linear combinations of the already calculated eigenvectors for the eigenvalue to obtain the 'correct' number of eigenvectors.

4 Assessment

To consider the effect of the graphics calculator on assessment, we need first to emphasise that assessment should be driven not by the available technology, but rather by the objectives incorporated in the mathematics syllabus. Valid testing and examining measures a student's ability to do the mathematics specified in the syllabus rather than merely execute calculator procedures. We consider here only the learning environment in which a graphics calculator is available for all teaching, learning, homework and assessment tasks and contrast such an environment with the present one.

The first consequence for assessment in mathematics, in tests and assignments as well as in examinations, is to de-emphasise routine drill and practice examples. A good examination, testing over a range of cognitive levels, will still contain drill and practice examples. However, with a graphics calculator, the nature of mathematical tasks is now different. Finding the value of a particular definite integral, a reasonably advanced task without a graphics calculator, is now routine. Marks allocated for successful completion of such a task can be expected to be reduced accordingly.

The reduction of instructional time on the development of manipulation skills is an opportunity for teachers to devote more time to applications. Students can now model phenomena such as the rainbow as described in section 2.1 that was previously beyond their experience. Traditionally both the formulation of the mathematical problem and the interpretation of the solution were seen by both teachers and students as of minor importance in comparison with solving the mathematical problem. With a graphics calculator, the 'solve the mathematical problem' stage of problem solving could become the least difficult one. Thus it can be anticipated that some assessment tasks will test only the student's skill in mathematical modelling, some only the interpretation of a mathematical solution, while some may test all three stages.

A graphics calculator requires new skills to be used effectively. These skills may be categorised [**?**] as numerical, graphical, symbolic and translation. As powerful as the numerical and graphical capabilities of the graphics calculator are, students need to recognise that many answers are approximations. Attention needs to be given to the precision limitations of the machine, particularly the cumulative effects of roundoff error. Assessment tasks need to state explicitly whether an exact answer is required or to what precision approximate answers need to be given. Clarifying the context of the question is the only way of determining which answers are worth full marks. In the context of doing mathematics rather than executing calculator procedures, and giving exact answers in the simplest form, neither "2.71828182846^X" nor “LN(e)*e^X” returned by the HP 38G is worth full marks as an answer to the question: ‘Differentiate $g(x) = e^x$.’

Assessment tasks need to be set to test how well students can interpret graphical information correctly, not just how well they can draw graphs. Thus in addition to the skills associated with a particular representation, numerical, graphical or symbolic, are the translation skills needed in moving from one representation to another. Assessment tasks need to be set to test these skills. Suitable example types are choosing the correct symbolic functional form given a graph, estimating a limit from a table of values or estimating the value of a definite integral from a graph.

With a de-emphasis on routine drill and practice examples, the future mathematics syllabi can be expected to have a greater emphasis on examples requiring students to interpret information or to give evidence of reasoning used in obtaining an answer. The following task tests students’ skills in interpreting and reasoning: ‘Let f be a function with positive values and let $g = 1/f$. If f has a local maximum at a what can you conclude about the function g and why?’

Finally, thought needs to be given not just to the questions comprising the assessment tasks but also to the marker’s role in evaluating student work. Marking keys commonly identify the individual steps necessary to solve the problem and maximum marks for each step are allocated accordingly. With the graphics calculator’s capacity for information storage, the question of how much working needs to be shown is a difficult one to answer but is discussed in [**?**].

The new skills required to do mathematics using a graphics calculator require a change in the tasks set to assess student achievement. Ensuring that assessment provides a valid measure of what a student knows, understands and can do will always be a challenge.

5 Summary

Graphics calculators have sparked a revolution in the teaching and examination of both secondary and tertiary mathematics. It is important that students learn to use the calculator intelligently to do mathematics rather than to merely press buttons. The graphics calculator is a very powerful tool. However, to use it effectively a student needs a good understanding of mathematics together with good algebraic and numeric awareness.