Mathematical Education in the Internet Era
— *Focussing on "Learning by Discovery" and an Experiment at ICME8/TG4* —

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Abstract

The internet has had many effects on mathematical education. Some teachers are already sending information or messages to students over the internet via email or web pages, and students are sending back their answers and other messages to their teachers over the internet. In this paper, we consider the best way of using the internet for distance learning of mathematics. We propose in this paper an efficient method of using computer algebra systems for a "learning by discovery" type distance education. The author of this paper carried out an experiment from Seville, Spain, in July, 1996, showing how efficient this method was, on the occasion of ICME8, at which he served as Chief Organiser of the Topic Group 4, "Learning Mathematics at a Distance".

1 Introduction

Once upon a time a few students would sit at the feet of a wise man, who would impart his wisdom to them in conversation. The teacher and students enjoyed a mutual respect, and knowledge was constructed as a result of their dialogue. Nowadays a university might have classes of 250, or even 500 students, and many of these do not listen to what their teacher is saying. Their minds are elsewhere during lectures, and for them the distance between teacher and student is virtually infinite.

However, over the past two decades the theories and practices associated with distance education have continually changed. At the present moment, with the influence of high technology profoundly affecting both theory and practice, it is difficult to recognise whether evolution has given way to revolution.

Nowadays, no matter what the physical separation between the two parties, if the learner is willing, and can study and communicate electronically — by email, TV, World Wide Web, etc — then the distance between student and teacher is once again effectively zero.

We are living through an information revolution; using modern technology our distance education system will become ever more sophisticated; in fact this is true of education as a whole. The style of education must change in line with these new opportunities, and if these are grasped then the future offers some very exciting prospects.

2 Manpower we need in the next century

In the current educational system, students are asked to learn materials given to them by teachers, solve problems in the class for a limited time and pass examinations, again in a given limited time. Thus students tend to try to memorize most of the materials, with or without understanding the theories behind them, classify the patterns of problems frequently given to them, and memorize methods for solving each of the patterns of problems. This is typical among Asian students in such countries as China, Korea, Japan and Singapore, where high school students have to pass severe entrance examination to universities. Thus, those students who consider the problem deeply from the beginning, who try to find out the secrets behind what they are seeing and understand the mathematical structure underlying there, no matter how much time this takes, would be left behind, and they would often find themselves among the students labelled "slow learners".

But these students are exactly those who will be needed in the next century. Thanks to the development of computers, those who can process routine work fast and accurate will not be needed in the future. Computers can do routine jobs far better than human beings, and the manpower we need in the next century will be those who have imagination and creativity. We have to train students along these lines. At least, we have to train them to have interest in things new to them, and encourage them to face to difficulties without fear, find out new facts and obtain the solution by themselves.

3 Learning by discovery

Until ten or fifteen years ago, computers were of no practical use in the mathematics education. We had to compute by hand, and this pencil and paper work was hard and time consuming. Thanks to the progress of computers, we can employ computers not only for numerical computation but also for symbolic manipulation. In the past, the ability of computing by hand quickly and accurately was very important, and inevitably we had to train students' skills along these lines. In this computer era, we may take it for granted that everyone is proficient in using the computer, and we have to reconsider what is more important to teach at schools.

Mathematics is originally a subject of experimental science in nature. As in physics or chemistry, eminent mathematicians like Euler or Gauss often found a simple rule governing the facts they had seen after a series of hard working complex and lengthy computational experiments. The effort of doing an experiment like this was tremendous in their time, for they had to do it by hand. But we now have a strong tool or weapon, namely, a computer. If we use a computer, it is neither difficult nor time-consuming to examine as many cases as we like, and contemporary mathematicians are using computers for their research in this way. Why not use this method for education? Instead of giving a lot of hard problems to be solved by pencil and paper in the classroom, we can show them some typical examples and/or problems using computer algebra systems such as Mathematica or Maple. Let students examine many cases by changing parameters or by rewriting text files given to them. After looking at many results obtained from the computer, a student would discover a mathematical rule underlying these, and could make a conjecture. Ask him/her to prove or disprove it. In this way, we can let the student not only learn the mathematics of the past but create his/her own mathematics. Students will be more interested in mathematics and will have a deeper understanding. We will call this methodology the method of "learning by discovery". It will enable us to take our education "from passive learning to active exploring".

4 The distance education system we propose

The above mentioned method of "learning by discovery" is most suitable for distance learning of mathematics in the internet era. There are a variety of methods of distance learning, from conventional correspondence by mail, through to the use of such high technology as satellite communications. However, all these methods have their drawbacks:– time delays in the case of mail, and time constraints and high cost (at both ends) in the case of satellite communications. Moreover, if one uses satellite broadcasting, the system will inevitably be essentially one-directional from teacher to students and it will be hard to create a bilateral environment. It should be noted that it is not always true that the more advanced the technology one uses the better teaching system one can develop.

The system we propose is as follows. Assume that a server is installed at the teaching centre and each learner has his/her own personal computer at hand, all of which are connected to the internet. By installing appropriate software such as Mathematica or Maple in both host (teacher) and terminal (student) machines, messages can be sent (cheaply and rapidly) in textfile form. Students are able to use the software to expand textfile commands into mathematical expressions, graphics, geometric figures etc., which require the transmission of prohibitive quantities of data if sent in "raw" form. One can install student version of Mathematica, say, which is not expensive, in the learner's machine. Write your teaching materials of "learning by discovery" type on your internet web pages, so that each learner can download it and play with them using the mathematical software installed in the terminal machine. If the teaching materials are similar to those written in ordinary text books, you may not be able to expect your students to learn mathematics by discovery. Therefore, designing your teaching materials carefully is very important.

You may ask whether the installation of mathematical software package onto the machine of individual learner is necessary or not. We assumed that all of our machines are connected to the internet, and therefore each learner can theoretically use the server's mathematical software package remotely. But if you do this, the speed of computation at each terminal will become very slow, especially when a learner is doing graphics. You can send a picture from the server as a bitmap image, but the information you have to send becomes very large and the time to send one picture will become very long. If a student is connecting his machine to the internet by modem, his/her telephone bill will become very high. On the other hand, if the system is designed as we proposed above, all we have to send are text files. After getting a text file, a student can disconnect his computer from the internet and spend as much time as he/she wishes to understand the content and solve problems. In the future we will have a more powerful network, the so-called information super-highway, connected by optical fibre. Then we may be able to send a big file in a short time. But then, the demand for sending big picture files will become greater and greater, and we will never be able to solve this traffic jam problem. It will be like the problem of constructing super-highways and the number of automobiles.

Open a bulletin board for question/answer and discussion, so that students can post problems of any nature. While each learner can ask questions to his/her teacher directly by email, he/she may prefer to post them to the bulletin board. It is well known among those involved in distance education that the "virtual campus" often generates more discussion between students than between student and teacher. Thus, creating networks of learners at a distance is important.

5 An experiment at ICME8/TG4

We carried out an experiment at ICME8. The aim of this experiment was to show how distance education can be enhanced by using computers with mathematical software packages connected to the internet. Immediately prior to the start of ICME, three problems were sent by Internet from Seville to students in several sites around the world. More than 50 answers were received, and in the second session of Topic Group 4, "Learning Mathematics at a Distance", these were discussed. For brevity we quote here just two of the problems and a few of the many responses to them.

[Problem 2] (i) You must know that the expression $x^2 + n$ ($n = 1, 2, 3, \ldots$) cannot be factorized into two expressions of the first order unless you use expressions with complex coefficients. Can you prove it?

(ii) What about the story for x^4+n ? Use Mathematica to compute Factor[x^4+1], Factor[x^4+2], Factor[x^4+3] and Factor[x^4+4]. You will find a surprising result. The expression x^4+4 can be factorized into two factors with integer coefficients. Is "4" an exceptional number having this property? Or are there many other numbers having this property? Write a suitable command for factorizing the expressions x^4+n for various n, and find out those n, if there are any, for which the expression can be factorized. (Remember that you are using a computer. Therefore, you can try cases for which n starts from 1 to 3000 or even to 6000 very easily.) If you find values of n, try to represent them in a general form. Furthermore, try to explain why they can be factorized for those specific values of n.

Answer (from a site in Singapore)

$n = 4, 64, 324, 1024, 2500, 5184, 9604, 16384, \ldots$ or $n = 2^2, 8^2, 18^2, 32^2, 50^2, 72^2, \ldots$ (Explanation: not given)

(From a site in Japan) We tried. The answer is

$$\begin{array}{ll}
n=4 & (2-2x+x^2)(2+2x+x^2) \\
n=64 & (8-4x+x^2)(8+4x+x^2) \\
n=324 & (18-6x+x^2)(18+6x+x^2) \\
n=1024 & (32-8x+x^2)(32+8x+x^2) \\
n=2500 & (50-10x+x^2)(50+10x+x^2) \\
n=5184 & (72-12x+x^2)(72+12x+x^2) \\
\ldots & \ldots
\end{array}$$

therefore we find that $n = 4 * m^4$ (m = integer).

Also, we found that the last digits of the sequence of above numbers $n = 4$, 64, 324, 1024, 2500, 5184, 9604, 16384, 26244, 40000, ..., for which x^4+n can be factorized, are 4, 4, 4, 4, 0, 4, 4, 4, 4, 0, 4, 4, 4, 4, 0, 4, ...

[Comment: It seems that they found a special case of Fermat's little theorem for the prime number 5].

[Problem 3] Factorize the expressions $x^n - 1$ for $n = 1, 2, 3, \ldots$ by using Mathematica, and observe the result.

Let us find any relation between n and factorization of $x^n - 1$. First, you will find that $x - 1$ is a factor of $x^n - 1$. Can you explain why? If you observe the result carefully, $x + 1$ is also a factor of $x^n - 1$ if and only if n is an even number. Again, can you explain why?

Try to find other facts which seem to be true for the factorization of $x^n - 1$. You do not need to prove the facts, but if you could, it would be excellent.

There may be patterns which first seem to be true but actually are false. Again, remember that you are using a computer. It will be easy for you to check your conjectures by testing large values of n (up to, say, $n = 500$).

Answer (from a site in Australia)

1. $x^n - 1$ has the same number of factors as the number of factors of n.
2. $x^{p-1} + x^{p-2} + x^{p-3} + ... + x^2 + x^1 + x^0$ is a factor of $x^n - 1$ when p is n's largest prime factor.
3. if n divides m then $(x^n - 1)$'s factors are a subset of $(x^m - 1)$'s factors.

(From a site in Singapore)

1. 1 is a root for $x^n - 1$.
2. When n is even, then $n = 2m$, $((-1)^2)^m = 1$, so -1 is a root.
3. conjecture: if $n > 2$ and n is a prime, then there is no other factor except $x - 1$.

(From a site in Japan)

It seems that if n is a prime number, then

$$x^n - 1 = (-1 + x)(1 + x + x^2 + ... + x^{n-1})$$

If n is not a prime, and if m is a factor of n, factor of $x^n - 1$ contains all the factors of $x^m - 1$. For example, we know that the factors of 12 are 2, 3, 4 and 6. For $n = 2$, 3, 4 and 6, we have

$$\begin{aligned} x^2 - 1 &= (-1 + x)(1 + x) \\ x^3 - 1 &= (-1 + x)(1 + x + x^2) \\ x^4 - 1 &= (-1 + x)(1 + x)(1 + x^2) \\ x^6 - 1 &= (-1 + x)(1 + x)(1 - x + x^2)(1 + x + x^2) \end{aligned}$$

And for $n = 12$ we see that

$$x^{12} - 1 = (-1 + x)(1 + x)(1 + x^2)(1 - x + x^2)(1 + x + x^2)(1 - x^2 + x^4)$$

The last factor is the only new factor appeared for $n = 12$. Also, one feature we found is that the coefficients seemed to be either $+1$ or -1.

But we later found a striking fact. Namely, for $n = 105$, we have a shocking result. For there are terms whose coefficient is 2 or -2. Same thing happens to $n = 165$. Actually, we have more 2's for $n = 165$.

It may have something to do with the fact that $105 = 3*5*7$ and $165 = 3*5*11$, but we do not know what it is. We still do not know if other numbers than 1 or 2 appear for coefficients.

[Comment: More numbers appear as coefficients for larger n.]

6 Conclusion

It is said that we are now in the information revolution era, and that our style of living will change in the next century. Electronic consuming is on the way, and digital money will be introduced sooner or later. The fact that we have already internet broadcasting and internet TV suggests that publishing, broadcasting and entertainment etc in the future will take different styles from those of today.

Education will not be an exception. Already, school education has been changed and improved by the use of computers. Using modern technology our distance education system will become ever more sophisticated, and anyone will be able to enjoy a better educational environment, no matter how remotely he/she lives.

However there are points we have to bear in mind. The question of getting help when 'stuck' on a particular mathematical point will be an issue. Such help must be available quickly if it is to be of use to the student. It may be more important to teach students a strategy for self-help. Students have different ways and patterns of learning, and a reflective strategy is both useful and important. One solution for these problems would be to create networks of learners at a distance, such as the bulletin board mentioned in section 4. Students will find it much easier to participate in such a venture if they have met 'face-to-face' beforehand. Though this is certainly desirable, it is probably unattainable when the distances involved are large.

Lack of access to the internet can disadvantage a student, but this will regrettably remain a possibility for some students for the foreseeable future. Bearing in mind the international nature of future distance education, it is also important to take account of the cultural background of the student.

It should be noted that the use of new technology can obscure important educational issues. There is a danger that concentrating on the media may distract from consideration of course content. Thus using the latest technology one might teach the mathematics of long ago, which would be inappropriate. All the educational institutions such as universities, high schools and government organisations in this region should evolve to make some use of distance teaching, and this process may help to clarify the situation.